

# Quaternion Algebra and Calculus

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This document provides a mathematical summary of quaternion algebra and calculus and how they relate to rotations and interpolation of rotations. The ideas are based on the article [1].

## 1 Quaternion Algebra

A *quaternion* is given by  $q = w + xi + yj + zk$  where  $w, x, y$ , and  $z$  are real numbers. Define  $q_n = w_n + x_n i + y_n j + z_n k$  ( $n = 0, 1$ ). *Addition* and *subtraction* of quaternions is defined by

$$\begin{aligned} q_0 \pm q_1 &= (w_0 + x_0 i + y_0 j + z_0 k) \pm (w_1 + x_1 i + y_1 j + z_1 k) \\ &= (w_0 \pm w_1) + (x_0 \pm x_1)i + (y_0 \pm y_1)j + (z_0 \pm z_1)k. \end{aligned} \quad (1)$$

Multiplication for the primitive elements  $i, j$ , and  $k$  is defined by  $i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ , and  $ki = -ik = j$ . *Multiplication* of quaternions is defined by

$$\begin{aligned} q_0 q_1 &= (w_0 + x_0 i + y_0 j + z_0 k)(w_1 + x_1 i + y_1 j + z_1 k) \\ &= (w_0 w_1 - x_0 x_1 - y_0 y_1 - z_0 z_1) + \\ &\quad (w_0 x_1 + x_0 w_1 + y_0 z_1 - z_0 y_1)i + \\ &\quad (w_0 y_1 - x_0 z_1 + y_0 w_1 + z_0 x_1)j + \\ &\quad (w_0 z_1 + x_0 y_1 - y_0 x_1 + z_0 w_1)k. \end{aligned} \quad (2)$$

Multiplication is not commutative in that the products  $q_0 q_1$  and  $q_1 q_0$  are not necessarily equal.

The *conjugate* of a quaternion is defined by

$$q^* = (w + xi + yj + zk)^* = w - xi - yj - zk. \quad (3)$$

The conjugate of a product of quaternions satisfies the properties  $(p^*)^* = p$  and  $(pq)^* = q^* p^*$ .

The *norm* of a quaternion is defined by

$$N(q) = N(w + xi + yj + zk) = w^2 + x^2 + y^2 + z^2. \quad (4)$$

The norm is a real-valued function and the norm of a product of quaternions satisfies the properties  $N(q^*) = N(q)$  and  $N(pq) = N(p)N(q)$ .

The *multiplicative inverse* of a quaternion  $q$  is denoted  $q^{-1}$  and has the property  $qq^{-1} = q^{-1}q = 1$ . It is constructed as

$$q^{-1} = q^* / N(q) \quad (5)$$

where the division of a quaternion by a real-valued scalar is just componentwise division. The inverse operation satisfies the properties  $(p^{-1})^{-1} = p$  and  $(pq)^{-1} = q^{-1}p^{-1}$ .

A simple but useful function is the *selection* function

$$W(q) = W(w + xi + yj + zk) = w \quad (6)$$

which selects the “real part” of the quaternion. This function satisfies the property  $W(q) = (q + q^*)/2$ .

The quaternion  $q = w + xi + yj + zk$  may also be viewed as  $q = w + \hat{v}$  where  $\hat{v} = xi + yj + zk$ . If we identify  $\hat{v}$  with the 3D vector  $(x, y, z)$ , then quaternion multiplication can be written using vector dot product ( $\bullet$ ) and cross product ( $\times$ ) as

$$(w_0 + \hat{v}_0)(w_1 + \hat{v}_1) = (w_0w_1 - \hat{v}_0 \bullet \hat{v}_1) + w_0\hat{v}_1 + w_1\hat{v}_0 + \hat{v}_0 \times \hat{v}_1. \quad (7)$$

In this form it is clear that  $q_0q_1 = q_1q_0$  if and only if  $\hat{v}_0 \times \hat{v}_1 = 0$  (these two vectors are parallel).

A quaternion  $q$  may also be viewed as a 4D vector  $(w, x, y, z)$ . The *dot product* of two quaternions is

$$q_0 \bullet q_1 = w_0w_1 + x_0x_1 + y_0y_1 + z_0z_1 = W(q_0q_1^*). \quad (8)$$

A *unit quaternion* is a quaternion  $q$  for which  $N(q) = 1$ . The inverse of a unit quaternion and the product of unit quaternions are themselves unit quaternions. A unit quaternion can be represented by

$$q = \cos \theta + \hat{u} \sin \theta \quad (9)$$

where  $\hat{u}$  as a 3D vector has length 1. However, observe that the quaternion product  $\hat{u}\hat{u} = -1$ . Note the similarity to unit length complex numbers  $\cos \theta + i \sin \theta$ . In fact, Euler's identity for complex numbers generalizes to quaternions,

$$\exp(\hat{u}\theta) = \cos \theta + \hat{u} \sin \theta, \quad (10)$$

where the exponential on the left-hand side is evaluated by symbolically substituting  $\hat{u}\theta$  into the power series representation for  $\exp(x)$  and replacing products  $\hat{u}\hat{u}$  by  $-1$ . From this identity it is possible to define the *power* of a unit quaternion,

$$q^t = (\cos \theta + \hat{u} \sin \theta)^t = \exp(\hat{u}t\theta) = \cos(t\theta) + \hat{u} \sin(t\theta). \quad (11)$$

It is also possible to define the *logarithm* of a unit quaternion,

$$\log(q) = \log(\cos \theta + \hat{u} \sin \theta) = \log(\exp(\hat{u}\theta)) = \hat{u}\theta. \quad (12)$$

It is important to note that the noncommutativity of quaternion multiplication disallows the standard identities for exponential and logarithm functions. The quaternions  $\exp(p)\exp(q)$  and  $\exp(p+q)$  are not necessarily equal. The quaternions  $\log(pq)$  and  $\log(p) + \log(q)$  are not necessarily equal.

## 2 Relationship of Quaternions to Rotations

A unit quaternion  $q = \cos \theta + \hat{u} \sin \theta$  represents the rotation of the 3D vector  $\hat{v}$  by an angle  $2\theta$  about the 3D axis  $\hat{u}$ . The rotated vector, represented as a quaternion, is  $R(\hat{v}) = q\hat{v}q^*$ . The proof requires showing that  $R(\hat{v})$  is a 3D vector, a length-preserving function of 3D vectors, a linear transformation, and does not have a reflection component.

To see that  $R(\hat{v})$  is a 3D vector,

$$\begin{aligned}
W(R(\hat{v})) &= W(q\hat{v}q^*) \\
&= [(q\hat{v}q^*) + (q\hat{v}q^*)^*]/2 \\
&= [q\hat{v}q^* + q\hat{v}^*q^*]/2 \\
&= q[(\hat{v} + \hat{v}^*)/2]q^* \\
&= qW(\hat{v})q^* \\
&= W(\hat{v}) \\
&= 0.
\end{aligned}$$

To see that  $R(\hat{v})$  is length-preserving,

$$\begin{aligned}
N(R(\hat{v})) &= N(q\hat{v}q^*) \\
&= N(q)N(\hat{v})N(q^*) \\
&= N(q)N(\hat{v})N(q) \\
&= N(\hat{v}).
\end{aligned}$$

To see that  $R(\hat{v})$  is a linear transformation, let  $a$  be a real-valued scalar and let  $\hat{v}$  and  $\hat{w}$  be 3D vectors; then

$$\begin{aligned}
R(a\hat{v} + \hat{w}) &= q(a\hat{v} + \hat{w})q^* \\
&= (qa\hat{v}q^*) + (q\hat{w}q^*) \\
&= a(q\hat{v}q^*) + (q\hat{w}q^*) \\
&= aR(\hat{v}) + R(\hat{w}),
\end{aligned}$$

thereby showing that the transform of a linear combination of vectors is the linear combination of the transforms.

The previous three properties show that  $R(\hat{v})$  is an orthonormal transformation. Such transformations include rotations *and* reflections. Consider  $R$  as a function of  $q$  for a *fixed vector*  $\hat{v}$ . That is,  $R(q) = q\hat{v}q^*$ . This function is a continuous function of  $q$ . For each  $q$  it is a linear transformation with determinant  $D(q)$ , so the determinant itself is a continuous function of  $q$ . Thus,  $\lim_{q \rightarrow 1} R(q) = R(1) = I$ , the identity function (the limit is taken along any path of quaternions which approach the quaternion 1) and  $\lim_{q \rightarrow 1} D(q) = D(1) = 1$ . By continuity,  $D(q)$  is identically 1 and  $R(q)$  does not have a reflection component.

Now we prove that the unit rotation axis is the 3D vector  $\hat{u}$  and the rotation angle is  $2\theta$ . To see that  $\hat{u}$  is a unit rotation axis we need only show that  $\hat{u}$  is unchanged by the rotation. Recall that  $\hat{u}^2 = \hat{u}\hat{u} = -1$ . This implies that  $\hat{u}^3 = -\hat{u}$ . Now

$$\begin{aligned}
R(\hat{u}) &= q\hat{u}q^* \\
&= (\cos \theta + \hat{u} \sin \theta)\hat{u}(\cos \theta - \hat{u} \sin \theta) \\
&= (\cos \theta)^2\hat{u} - (\sin \theta)^2\hat{u}^3 \\
&= (\cos \theta)^2\hat{u} - (\sin \theta)^2(-\hat{u}) \\
&= \hat{u}.
\end{aligned}$$

To see that the rotation angle is  $2\theta$ , let  $\hat{u}$ ,  $\hat{v}$ , and  $\hat{w}$  be a right-handed set of orthonormal vectors. That is, the vectors are all unit length;  $\hat{u} \bullet \hat{v} = \hat{u} \bullet \hat{w} = \hat{v} \bullet \hat{w} = 0$ ; and  $\hat{u} \times \hat{v} = \hat{w}$ ,  $\hat{v} \times \hat{w} = \hat{u}$ , and  $\hat{w} \times \hat{u} = \hat{v}$ . The vector  $\hat{v}$  is rotated by an angle  $\phi$  to the vector  $q\hat{v}q^*$ , so  $\hat{v} \bullet (q\hat{v}q^*) = \cos(\phi)$ . Using equation (8) and  $\hat{v}^* = -\hat{v}$ , and  $\hat{p}^2 = -1$  for unit quaternions with zero real part,

$$\begin{aligned}
\cos(\phi) &= \hat{v} \bullet (q\hat{v}q^*) \\
&= W(\hat{v}^*q\hat{v}q^*) \\
&= W[-\hat{v}(\cos\theta + \hat{u}\sin\theta)\hat{v}(\cos\theta - \hat{u}\sin\theta)] \\
&= W[(-\hat{v}\cos\theta - \hat{v}\hat{u}\sin\theta)(\hat{v}\cos\theta - \hat{v}\hat{u}\sin\theta)] \\
&= W[-\hat{v}^2(\cos\theta)^2 + \hat{v}^2\hat{u}\sin\theta\cos\theta - \hat{v}\hat{u}\hat{v}\sin\theta\cos\theta + (\hat{v}\hat{u})^2(\sin\theta)^2] \\
&= W[(\cos\theta)^2 - (\sin\theta)^2 - (\hat{u} + \hat{v}\hat{u}\hat{v})\sin\theta\cos\theta]
\end{aligned}$$

Now  $\hat{v}\hat{u} = -\hat{v} \bullet \hat{u} + \hat{v} \times \hat{u} = \hat{v} \times \hat{u} = -\hat{w}$  and  $\hat{v}\hat{u}\hat{v} = -\hat{w}\hat{v} = \hat{w} \bullet \hat{v} - \hat{w} \times \hat{v} = \hat{u}$ . Consequently,

$$\begin{aligned}
\cos(\phi) &= W[(\cos\theta)^2 - (\sin\theta)^2 - (\hat{u} + \hat{v}\hat{u}\hat{v})\sin\theta\cos\theta] \\
&= W[(\cos\theta)^2 - (\sin\theta)^2 - \hat{u}(2\sin\theta\cos\theta)] \\
&= (\cos\theta)^2 - (\sin\theta)^2 \\
&= \cos(2\theta)
\end{aligned}$$

and the rotation angle is  $\phi = 2\theta$ .

It is important to note that the quaternions  $q$  and  $-q$  represent the same rotation since  $(-q)\hat{v}(-q)^* = q\hat{v}q^*$ . While either quaternion will do, the interpolation methods require choosing one over the other.

### 3 Quaternion Calculus

The only support we need for quaternion interpolation is to differentiate unit quaternion functions raised to a real-valued power. These formulas are identical to those derived in a standard calculus course, but the order of multiplication must be observed.

The derivative of the function  $q^t$  where  $q$  is a constant unit quaternion is

$$\frac{d}{dt}q^t = q^t \log(q) \tag{13}$$

where  $\log$  is the function defined earlier by  $\log(\cos\theta + \hat{u}\sin\theta) = \hat{u}\theta$ . To prove this, observe that

$$q^t = \cos(t\theta) + \hat{u}\sin(t\theta)$$

in which case

$$\frac{d}{dt}q^t = -\sin(t\theta)\theta + \hat{u}\cos(t\theta)\theta = \hat{u}\hat{u}\sin(t\theta)\theta + \hat{u}\cos(t\theta)\theta$$

where we have used  $-1 = \hat{u}\hat{u}$ . Factoring this, we have

$$\frac{d}{dt}q^t = (\hat{u}\sin(t\theta) + \cos(t\theta))\hat{u}\theta = q^t \log(q)$$

The right-hand side also factors as  $\log(q)q^t$ . Generally, the order of operands in a quaternion multiplication is important, but not in this special case. The power can be a function itself,

$$\frac{d}{dt}q^{f(t)} = f'(t)q^{f(t)}\log(q) \quad (14)$$

The method of proof is the same as that of the previous case where  $f(t) = t$ .

Generally, a quaternion function may be written as

$$q(t) = \cos(\theta(t)) + \hat{u}(t)\sin(\theta(t)) \quad (15)$$

where the angle  $\theta$  and  $\hat{u}$  both vary with  $t$ . The derivative is

$$q'(t) = -\sin(\theta(t))\theta'(t) + \hat{u}(t)\cos(\theta(t))\theta'(t) + \hat{u}'(t)\sin(\theta(t)) = q(t)\hat{u}(t)\theta'(t) + \hat{u}'(t)\sin(\theta(t)) \quad (16)$$

Because  $-1 = \hat{u}(t)\hat{u}(t)$ , we also know that

$$0 = \hat{u}(t)\hat{u}'(t) + \hat{u}'(t)\hat{u}(t) \quad (17)$$

If you write  $\hat{u} = xi + yj + zk$  and expand the right-hand side of Equation (17), the equation becomes  $xx' + yy' + zz' = 0$ . This implies the vectors  $\mathbf{u} = (x, y, z)$  and  $\mathbf{u}' = (x', y', z')$  are perpendicular. From this discussion, it is easily shown that

$$\hat{u}(t)q'(t) + q'(t)\hat{u}(t) = -2\theta'(t)q(t) \quad (18)$$

Now define

$$h(t) = q(t)^{f(t)} = \cos(f(t)\theta(t)) + \hat{u}(t)\sin(f(t)\theta(t)) \quad (19)$$

where  $q(t) = \cos(\theta(t)) + \hat{u}(t)\sin(\theta(t))$ . The motivation for the definition is that we know how to compute  $q(t)^{f(s)}$  for independent variables  $s$  and  $t$ , and we want this to be jointly continuous in the sense that  $q(t)^{f(t)} = \lim_{s \rightarrow t} q(t)^{f(s)}$ . The derivative is

$$h'(t) = [-\sin(f\theta) + \hat{u}\cos(f\theta)](f\theta)' + \hat{u}'\sin(f\theta) = (\hat{u}h)(f\theta)' + \hat{u}'\sin(f\theta) \quad (20)$$

Similar to Equation (18), it may be shown that

$$\hat{u}(t)h'(t) + h'(t)\hat{u}(t) = -2\frac{d[f(t)\theta(t)]}{dt}h(t)$$

Note that this last equation by itself is not enough information to completely determine  $h'(t)$ , so consider it a sufficient condition for the derivative  $h'(t)$ .

## 4 Spherical Linear Interpolation

The previous version of the document had a construction for spherical linear interpolation of two quaternions  $q_0$  and  $q_1$  treated as unit length vectors in 4-dimensional space, the angle  $\theta$  between them acute. The idea was that  $q(t) = c_0(t)q_0 + c_1(t)q_1$  where  $c_0(t)$  and  $c_1(t)$  are real-valued functions for  $0 \leq t \leq 1$  with  $c_0(0) = 1$ ,  $c_1(0) = 0$ ,  $c_0(1) = 0$ , and  $c_1(1) = 1$ . The quantity  $q(t)$  is required to be a unit vector, so  $1 = q(t) \bullet q(t) = c_0(t)^2 + 2\cos(\theta)c_0(t)c_1(t) + c_1(t)^2$ . This is the equation of an ellipse that I factored using methods of analytic geometry to obtain formulas for  $c_0(t)$  and  $c_1(t)$ .

A simpler construction uses only trigonometry and solving two equations in two unknowns. As  $t$  uniformly varies between 0 and 1, the values  $q(t)$  are required to uniformly vary along the circular arc from  $q_0$  to  $q_1$ . That is, the angle between  $q(t)$  and  $q_0$  is  $\cos(t\theta)$  and the angle between  $q(t)$  and  $q_1$  is  $\cos((1-t)\theta)$ . Dotting the equation for  $q(t)$  with  $q_0$  yields

$$\cos(t\theta) = c_0(t) + \cos(\theta)c_1(t)$$

and dotting the equation with  $q_1$  yields

$$\cos((1-t)\theta) = \cos(\theta)c_0(t) + c_1(t).$$

These are two equations in the two unknowns  $c_0$  and  $c_1$ . The solution for  $c_0$  is

$$c_0(t) = \frac{\cos(t\theta) - \cos(\theta)\cos((1-t)\theta)}{1 - \cos^2(\theta)} = \frac{\sin((1-t)\theta)}{\sin(\theta)}.$$

The last equality is obtained by applying double-angle formulas for sine and cosine. By symmetry,  $c_1(t) = c_0(1-t)$ . Or solve the equations for

$$c_1(t) = \frac{\cos((1-t)\theta) - \cos(\theta)\cos(t\theta)}{1 - \cos^2(\theta)} = \frac{\sin(t\theta)}{\sin(\theta)}.$$

The spherical linear interpolation, abbreviated as *slerp*, is defined by

$$\text{Slerp}(t; q_0, q_1) = \frac{q_0 \sin((1-t)\theta) + q_1 \sin(t\theta)}{\sin \theta} \quad (21)$$

for  $0 \leq t \leq 1$ .

Although  $q_1$  and  $-q_1$  represent the same rotation, the values of  $\text{Slerp}(t; q_0, q_1)$  and  $\text{Slerp}(t; q_0, -q_1)$  are not the same. It is customary to choose the sign  $\sigma$  on  $q_1$  so that  $q_0 \bullet (\sigma q_1) \geq 0$  (the angle between  $q_0$  and  $\sigma q_1$  is acute). This choice avoids extra spinning caused by the interpolated rotations.

For unit quaternions, *slerp* can be written as

$$\text{Slerp}(t; q_0, q_1) = q_0 (q_0^{-1} q_1)^t. \quad (22)$$

The idea is that  $q_1 = q_0(q_0^{-1}q_1)$ . The term  $q_0^{-1}q_1 = \cos \theta + \hat{u} \sin \theta$  where  $\theta$  is the angle between  $q_0$  and  $q_1$ . The time parameter can be introduced into the angle so that the adjustment of  $q_0$  varies uniformly with over the great arc between  $q_0$  and  $q_1$ . That is,  $q(t) = q_0[\cos(t\theta) + \hat{u} \sin(t\theta)] = q_0[\cos \theta + \hat{u} \sin \theta]^t = q_0(q_0^{-1}q_1)^t$ .

The derivative of *slerp* in the form of equation (22) is a simple application of equation (13),

$$\text{Slerp}'(t; q_0, q_1) = q_0(q_0^{-1}q_1)^t \log(q_0^{-1}q_1). \quad (23)$$

## 5 Spherical Cubic Interpolation

Cubic interpolation of quaternions can be achieved using a method described in [2] which has the flavor of bilinear interpolation on a quadrilateral. The evaluation uses an iteration of three *slerps* and is similar to the de Casteljau algorithm (see [3]). Imagine four quaternions  $p$ ,  $a$ ,  $b$ , and  $q$  as the ordered vertices of a quadrilateral. Interpolate  $c$  along the “edge” from  $p$  to  $q$  using *slerp*. Interpolate  $d$  along the “edge” from

$a$  to  $b$ . Now interpolate the edge interpolations  $c$  and  $d$  to get the final result  $e$ . The end result is denoted *squad* and is given by

$$\text{Squad}(t; p, a, b, q) = \text{Slerp}(2t(1-t); \text{Slerp}(t; p, q), \text{Slerp}(t; a, b)) \quad (24)$$

For unit quaternions we can use equation (22) to obtain a similar formula for *squad*,

$$\text{Squad}(t; p, a, b, q) = \text{Slerp}(t; p, q)(\text{Slerp}(t; p, q)^{-1} \text{Slerp}(t; a, b))^{2t(1-t)} \quad (25)$$

The derivative of *squad* in equation (25) is computed as follows. To simplify the notation, define  $U(t) = \text{Slerp}(t; p, q)$  and  $V(t) = \text{Slerp}(t; a, b)$ . Equation (13) implies  $U'(t) = U(t) \log(p^{-1}q)$  and  $V'(t) = V(t) \log(a^{-1}b)$ . Define  $W(t)$ ,  $\hat{\alpha}(t)$ , and  $\phi(t)$  by

$$W(t) = U(t)^{-1}V(t) = \cos(\phi(t)) + \hat{\alpha}(t) \sin(\phi(t)) \quad (26)$$

It is simple to see that  $U(t)W(t) = V(t)$ . The derivative of  $W(t)$  is implicit in  $U(t)W'(t) + U'(t)W(t) = V'(t)$ . The *squad* function is then

$$\text{Squad}(t; p, a, b, q) = U(t)W(t)^{2t(1-t)} \quad (27)$$

and its derivative is computed as shown next, using Equation (20),

$$\begin{aligned} \text{Squad}'(t; p, q, a, b) &= \frac{d}{dt} [UW^{2t(1-t)}] \\ &= U(t) \frac{d}{dt} [W(t)^{2t(1-t)}] + U'(t) [W(t)^{2t(1-t)}] \\ &= U(t) \left\{ \hat{\alpha}(t)W(t)^{2t(1-t)} [2t(1-t)\phi'(t) + (2-4t)\phi(t)] + \hat{\alpha}'(t) \sin(2t(1-t)\phi(t)) \right\} \\ &\quad + U'(t)W(t)^{2t(1-t)} \end{aligned} \quad (28)$$

For spline interpolation using *squad* we will need to evaluate the derivative of *squad* at  $t = 0$  and  $t = 1$ . Observe that  $U(0) = p$ ,  $U'(0) = p \log(p^{-1}q)$ ,  $U(1) = q$ ,  $U'(1) = q \log(p^{-1}q)$ ,  $V(0) = a$ ,  $V'(0) = a \log(a^{-1}b)$ ,  $V(1) = b$ , and  $V'(1) = b \log(a^{-1}b)$ . Also observe that  $\log(W(t)) = \hat{\alpha}(t)\phi(t)$  so that  $\log(p^{-1}a) = \log(W(0)) = \hat{\alpha}(0)\phi(0)$  and  $\log(q^{-1}b) = \log(W(1)) = \hat{\alpha}(1)\phi(1)$ . The derivatives of *squad* at the endpoints are

$$\begin{aligned} \text{Squad}'(0; p, a, b, q) &= U(0) \{ \hat{\alpha}(0)[+2\phi(0)] \} + U'(0) = p[\log(p^{-1}q) + 2\log(p^{-1}a)] \\ \text{Squad}'(1; p, a, b, q) &= U(1) \{ \hat{\alpha}(1)[-2\phi(1)] \} + U'(1) = q[\log(p^{-1}q) - 2\log(q^{-1}b)] \end{aligned} \quad (29)$$

## 6 Spline Interpolation of Quaternions

Given a sequence of  $N$  unit quaternions  $\{q_n\}_{n=0}^{N-1}$ , we want to build a spline which interpolates those quaternions subject to the conditions that the spline pass through the control points and that the derivatives are continuous. The idea is to choose intermediate quaternions  $a_n$  and  $b_n$  to allow control of the derivatives at the endpoints of the spline segments. More precisely, let  $S_n(t) = \text{Squad}(t; q_n, a_n, b_{n+1}, q_{n+1})$  be the spline segments. By definition of *squad* it is easily shown that

$$S_{n-1}(1) = q_n = S_n(0).$$



To obtain continuous derivatives at the endpoints we need to match the derivatives of two consecutive spline segments,

$$S'_{n-1}(1) = S'_n(0).$$

It can be shown from equation (29) that

$$S'_{n-1}(1) = q_n[\log(q_{n-1}^{-1}q_n) - 2\log(q_n^{-1}b_n)]$$

and

$$S'_n(0) = q_n[\log(q_n^{-1}q_{n+1}) + 2\log(q_n^{-1}a_n)].$$

The derivative continuity equation provides one equation in the two unknowns  $a_n$  and  $b_n$ , so we have one degree of freedom. As suggested in [1], a good choice for the derivative at the control point uses an average  $T_n$  of “tangents”, so  $S'_{n-1}(1) = q_n T_n = S'_n(0)$  where

$$T_n = \frac{\log(q_n^{-1}q_{n+1}) + \log(q_{n-1}^{-1}q_n)}{2}. \quad (30)$$

We now have two equations to determine  $a_n$  and  $b_n$ . Some algebra will show that

$$a_n = b_n = q_n \exp\left(-\frac{\log(q_n^{-1}q_{n+1}) + \log(q_{n-1}^{-1}q_n)}{4}\right). \quad (31)$$

Thus,  $S_n(t) = \text{Squad}(t; q_n, a_n, a_{n+1}, q_{n+1})$ .

EXAMPLE. To illustrate the cubic nature of the interpolation, consider a sequence of quaternions whose general term is  $q_n = \exp(i\theta_n)$ . This is a sequence of complex numbers whose products do commute and for which the usual properties of exponents and logarithms do apply. The intermediate terms are

$$a_n = \exp(-i(\theta_{n+1} - 6\theta_n + \theta_{n-1})/4).$$

Also,

$$\text{Slerp}(t, q_n, q_{n+1}) = \exp(i((1-t)\theta_n + t\theta_{n+1}))$$

and

$$\text{Slerp}(t, a_n, a_{n+1}) = \exp(-i((1-t)(\theta_{n+1} - 6\theta_n + \theta_{n-1}) + t(\theta_{n+2} - 6\theta_{n+1} + \theta_n))/4).$$

Finally,

$$\begin{aligned} \text{Squad}(t, q_n, a_n, a_{n+1}, q_{n+1}) &= \exp([1 - 2t(1-t)][(1-t)\theta_n + t\theta_{n+1}] \\ &\quad - [2t(1-t)/4][(1-t)(\theta_{n+1} - 6\theta_n + \theta_{n-1}) + t(\theta_{n+2} - 6\theta_{n+1} + \theta_n)]). \end{aligned}$$

The angular cubic interpolation is

$$\phi(t) = -\frac{1}{2}t^2(1-t)\theta_{n+2} + \frac{1}{2}t(2 + 2(1-t) - 3(1-t)^2)\theta_{n+1} + \frac{1}{2}(1-t)(2 + 2t - 3t^2)\theta_n - \frac{1}{2}t(1-t)^2\theta_{n-1}.$$

It can be shown that  $\phi(0) = \theta_n$ ,  $\phi(1) = \theta_{n+1}$ ,  $\phi'(0) = (\theta_{n+1} - \theta_{n-1})/2$ , and  $\phi'(1) = (\theta_{n+2} - \theta_n)/2$ . The derivatives at the end points are centralized differences, the average of left and right derivatives as expected.

## References

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