Perspective Mappings Between Cuboids

David Eberly
Geometric Tools, LLC
http://www.geometrictools.com/

Created: March 2, 1999
Last Modified: March 1, 2008

Contents

1 Mapping Unit Square to Quadrilateral

The problem of mapping the unit square to a quadrilateral can be solved by considering the problem in a three dimensional setting. Translate one vertex of the quadrilateral to the origin (say this is $\mathbf{q}_{00}$). Label the other vertices in counterclockwise order as $\mathbf{q}_{10}$, $\mathbf{q}_{11}$, and $\mathbf{q}_{01}$. The plane containing these points is $z = 0$ and has normal $(0, 0, 1)$. Select an eye point $\mathbf{E} = (e_0, e_1, e_2)$. Rotate the plane of the quadrilateral so that its normal is $\mathbf{N} = (n_0, n_1, n_2)$. The quadrilateral can be projected onto the viewing plane $z = 0$ by a perspective projection. The idea is to choose an eye point $\mathbf{E}$ and normal $\mathbf{N}$ so that the projection is the unit square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$, and $\boldsymbol{\xi} = (x, y, 0)$. The perspective mapping involves finding the intersection of the line $(1 - t)\boldsymbol{\xi} + t\mathbf{E}$ with the plane $\mathbf{N} \cdot \boldsymbol{\xi} = 0$. Replacing the line equation in the plane equation, solving for t, the mapping is

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{(\mathbf{N} \cdot \boldsymbol{\xi})\mathbf{E} - (\mathbf{N} \cdot \mathbf{E})\boldsymbol{\xi}}{\mathbf{N} \cdot (\boldsymbol{\xi} - \mathbf{E})}.$$

The four rays through the quadrilateral points on the plane $\mathbf{N} \cdot (x, y, z) = 0$ must intersect the four corners of the square. Let $\mathbf{p}_{10}$, $\mathbf{p}_{01}$, and $\mathbf{p}_{11}$ be the four quadrilateral points in that plane. Then

$$\begin{aligned}
\mathbf{p}_{10} &= \frac{(\mathbf{N}\cdot\mathbf{i})\mathbf{E} - (\mathbf{N}\cdot\mathbf{E})\mathbf{i}}{\mathbf{N}\cdot(\mathbf{i}-\mathbf{E})} \\
\mathbf{p}_{01} &= \frac{(\mathbf{N}\cdot\mathbf{j})\mathbf{E} - (\mathbf{N}\cdot\mathbf{E})\mathbf{j}}{\mathbf{N}\cdot(\mathbf{j}-\mathbf{E})} \\
\mathbf{p}_{11} &= \frac{(\mathbf{N}\cdot(\mathbf{i}+\mathbf{j}))\mathbf{E} - (\mathbf{N}\cdot\mathbf{E})(\mathbf{i}+\mathbf{j})}{\mathbf{N}\cdot(\mathbf{i}+\mathbf{j}-\mathbf{E})}
\end{aligned}$$

Also, $\mathbf{p}_{11} = \alpha\mathbf{p}_{10} + \beta\mathbf{p}_{01}$. The equation for $\mathbf{p}_{11}$ can be solved for α and β using the two equations for $\mathbf{p}_{10}$ and $\mathbf{p}_{01}$:

$$\begin{aligned}
\alpha &= \frac{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i})}{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i}-\mathbf{j})} \\
\beta &= \frac{\mathbf{N}\cdot(\mathbf{E}-\mathbf{j})}{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i}-\mathbf{j})}
\end{aligned}$$

The equations for α and β can be rewritten as

$$\begin{aligned}
\mathbf{N} \cdot ((\alpha - 1)(\mathbf{E} - \mathbf{i}) - \alpha\mathbf{j}) = 0 \\
\mathbf{N} \cdot ((\beta - 1)(\mathbf{E} - \mathbf{j}) - \beta\mathbf{i}) = 0
\end{aligned}$$

Vector $\mathbf{N}$ may be selected as the cross product of the two vectors which are perpendicular to it,

$$\mathbf{N} = ((\beta - 1)e_2, (\alpha - 1)e_2, (\alpha + \beta - 1) + (1 - \beta)e_0 + (1 - \alpha)e_1)\,.$$

Consequently,

$$\begin{aligned}
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i}) &= \alpha e_2 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{j}) &= \beta e_2 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i} - \mathbf{j}) &= e_2 \\
\mathbf{N} \cdot \mathbf{E} &= (\alpha + \beta - 1)e_2
\end{aligned}$$

The general mapping is

$$\begin{aligned}
\mathbf{p} &= \frac{[\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j})]\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})(x\mathbf{i}+y\mathbf{j})}{\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}-\mathbf{E})} \\
&= \frac{x[(\mathbf{N}\cdot\mathbf{i})\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})\mathbf{i}]+y[(\mathbf{N}\cdot\mathbf{j})\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})\mathbf{j}]}{\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}-\mathbf{E})} \\
&= \frac{x\mathbf{N}\cdot(\mathbf{i}-\mathbf{E})\mathbf{p}_{10}+y\mathbf{N}\cdot(\mathbf{j}-\mathbf{E})\mathbf{p}_{01}}{\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}-\mathbf{E})} \\
&= \frac{\alpha x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}\mathbf{p}_{10} + \frac{\beta x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}\mathbf{p}_{01}.
\end{aligned}$$

Rotating the plane back to $z = 0$ and translating the origin back to the original vertex, the mapping is

$$\mathbf{q}-\mathbf{q}_{00} = \frac{\alpha x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}(\mathbf{q}_{10}-\mathbf{q}_{00}) + \frac{\beta x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}(\mathbf{q}_{01}-\mathbf{q}_{00}).$$

2 Mapping Unit Cube to Cuboid

The problem of mapping the unit cube to a cuboid can be solved by considering the problem in a four dimensional setting (coordinates are (x, y, z, w)). Translate one vertex of the cuboid to the origin (say this is $\mathbf{q}_{000}$). Label the other vertices as $\mathbf{q}_{100}$, $\mathbf{q}_{010}$, $\mathbf{q}_{001}$, $\mathbf{q}_{110}$, $\mathbf{q}_{101}$, $\mathbf{q}_{011}$, and $\mathbf{q}_{111}$. The ordering of the vertices corresponds to the ordering of those in the cube whose vertices are the subscripts of the $\mathbf{q}$ values. The hyperplane containing these points is $z = 0$ and has normal $(0, 0, 0, 1)$. Select an eye point $\mathbf{E} = (e_0, e_1, e_2, e_3)$. Rotate the hyperplane of the cuboid so that its normal is $\mathbf{N} = (n_0, n_1, n_2, n_3)$. The cuboid can be projected onto the viewing volume $w = 0$ by a perspective projection. The idea is to choose an eye point $\mathbf{E}$ and normal $\mathbf{N}$ so that the projection is the unit cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$.

Let $\mathbf{i} = (1, 0, 0, 0)$, $\mathbf{j} = (0, 1, 0, 0)$, $\mathbf{k} = (0, 0, 1, 0)$, $\boldsymbol{\ell} = (0, 0, 0, 1)$, and $\boldsymbol{\xi} = (x, y, z, 0)$. The perspective mapping involves finding the intersection of the line $(1-t)\boldsymbol{\xi}+t\mathbf{E}$ with the plane $\mathbf{N}\cdot\boldsymbol{\xi} = 0$. Replacing the line equation in the plane equation, solving for t, the mapping is

$$(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = \frac{(\mathbf{N}\cdot\boldsymbol{\xi})\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})\boldsymbol{\xi}}{\mathbf{N}\cdot(\boldsymbol{\xi}-\mathbf{E})}.$$

The eight rays through the cuboid points on the hyperplane $\mathbf{N}\cdot(x, y, z, w) = 0$ must intersect the eight corners of the cube. Let $\mathbf{p}_{ijk}$ be the eight cuboid points in the hyperplane. Then

$$\mathbf{p}_{ijk} = \frac{(\mathbf{N}\cdot\mathbf{b}_{ijk})\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})\mathbf{b}_{ijk}}{\mathbf{N}\cdot(\mathbf{b}_{ijk}-\mathbf{E})}$$

where $\mathbf{b}_{ijk}$ are the eight vertices of the cube.

Also, $\mathbf{p}_{111} = \alpha\mathbf{p}_{100} + \beta\mathbf{p}_{010} + \gamma\mathbf{p}_{001}$. This equations can be solved using the other equations

$$\begin{aligned}
\alpha &= \frac{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i})}{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i}-\mathbf{j}-\mathbf{k})} \\
\beta &= \frac{\mathbf{N}\cdot(\mathbf{E}-\mathbf{j})}{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i}-\mathbf{j}-\mathbf{k})} \\
\gamma &= \frac{\mathbf{N}\cdot(\mathbf{E}-\mathbf{k})}{\mathbf{N}\cdot(\mathbf{E}-\mathbf{i}-\mathbf{j}-\mathbf{k})}
\end{aligned}$$

The equations for α, β, and γ can be rewritten as

$$\begin{aligned}
\mathbf{N} \cdot ((\alpha - 1)(\mathbf{E} - \mathbf{i}) - \alpha\mathbf{j} - \alpha\mathbf{k} = 0 \\
\mathbf{N} \cdot ((\beta - 1)(\mathbf{E} - \mathbf{j}) - \beta\mathbf{i} - \beta\mathbf{k}) = 0 \\
\mathbf{N} \cdot ((\gamma - 1)(\mathbf{E} - \mathbf{k}) - \gamma\mathbf{i} - \gamma\mathbf{j} = 0
\end{aligned}$$

Vector $\mathbf{N}$ may be selected as the generalized cross product of the three vectors which are perpendicular to it,

$$\begin{aligned}
n_0 &= (-1 - \alpha + \beta + \gamma)e_3, \\
n_1 &= (-1 + \alpha - \beta + \gamma)e_3, \\
n_2 &= (-1 + \alpha + \beta - \gamma)e_3, \\
n_3 &= (\alpha + \beta + \gamma - 1) + (1 + \alpha - \beta - \gamma)e_0 + (1 - \alpha + \beta - \gamma)e_1 + (1 - \alpha - \beta + \gamma)e_2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i}) &= 2\alpha e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{j}) &= 2\beta e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{k}) &= 2\gamma e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i} - \mathbf{j}) &= (1 + \alpha + \beta - \gamma)e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i} - \mathbf{k}) &= (1 + \alpha - \beta + \gamma)e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{j} - \mathbf{k}) &= (1 - \alpha + \beta + \gamma)e_3 \\
\mathbf{N} \cdot (\mathbf{E} - \mathbf{i} - \mathbf{j} - \mathbf{k}) &= 2e_3 \\
\mathbf{N} \cdot \mathbf{E} &= (\alpha + \beta + \gamma - 1)e_3
\end{aligned}$$

Because the object is a cuboid, there are some restrictions on its vertices. If the other vertices are defined as

$$\begin{aligned}
\mathbf{p}_{110} &= v_0\mathbf{p}_{100} + v_1\mathbf{p}_{010} \\
\mathbf{p}_{101} &= u_0\mathbf{p}_{100} + u_2\mathbf{p}_{001} \\
\mathbf{p}_{011} &= w_1\mathbf{p}_{010} + w_2\mathbf{p}_{001}
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{N} \cdot ((v_0 - 1)(\mathbf{E} - \mathbf{i}) - v_0\mathbf{j}) = 0 \\
\mathbf{N} \cdot ((v_1 - 1)(\mathbf{E} - \mathbf{j}) - v_1\mathbf{i}) = 0 \\
\mathbf{N} \cdot ((u_0 - 1)(\mathbf{E} - \mathbf{i}) - u_0\mathbf{k}) = 0 \\
\mathbf{N} \cdot ((u_2 - 1)(\mathbf{E} - \mathbf{k}) - u_2\mathbf{i}) = 0 \\
\mathbf{N} \cdot ((w_1 - 1)(\mathbf{E} - \mathbf{j}) - w_1\mathbf{k}) = 0 \\
\mathbf{N} \cdot ((w_2 - 1)(\mathbf{E} - \mathbf{k}) - w_2\mathbf{j}) = 0
\end{aligned}$$

which implies the conditions

$$\begin{aligned} v_0 &= \tfrac{2\alpha}{1+\alpha+\beta-\gamma} \\ v_1 &= \tfrac{2\beta}{1+\alpha+\beta-\gamma} \\ u_0 &= \tfrac{2\alpha}{1+\alpha-\beta+\gamma} \\ u_2 &= \tfrac{2\gamma}{1+\alpha-\beta+\gamma} \\ w_1 &= \tfrac{2\beta}{1-\alpha+\beta+\gamma} \\ w_2 &= \tfrac{2\gamma}{1-\alpha+\beta+\gamma} \end{aligned}$$

The general mapping is

$$\begin{aligned} \mathbf{p} &= \frac{[\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]\mathbf{E}-(\mathbf{N}\cdot\mathbf{E})(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}-\mathbf{E})} \\ &= \frac{x\mathbf{N}\cdot(\mathbf{i}-\mathbf{E})\mathbf{p}_{100}+y\mathbf{N}\cdot(\mathbf{j}-\mathbf{E})\mathbf{p}_{010}+z\mathbf{N}\cdot(\mathbf{k}-\mathbf{E})\mathbf{p}_{001}}{\mathbf{N}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}-\mathbf{E})} \\ &= \tfrac{2\alpha x}{\Delta}\mathbf{p}_{100} + \tfrac{2\beta y}{\Delta}\mathbf{p}_{010} + \tfrac{2\gamma z}{\Delta}\mathbf{p}_{001} \end{aligned}$$

where

$$\Delta = (\alpha+\beta+\gamma-1) + (1+\alpha-\beta-\gamma)x + (1-\alpha+\beta-\gamma)y + (1-\alpha-\beta+\gamma)z$$

Rotating the plane back to $z = 0$ and translating the origin back to the original vertex, the mapping is

$$\mathbf{q} - \mathbf{q}_{000} = \frac{2\alpha x}{\Delta}(\mathbf{q}_{100} - \mathbf{q}_{000}) + \frac{2\beta y}{\Delta}(\mathbf{q}_{010} - \mathbf{q}_{000}) + \frac{2\gamma z}{\Delta}(\mathbf{q}_{001} - \mathbf{q}_{000}).$$