Centers of a Simplex

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Let the vertices of an n-dimensional simplex be $\mathbf{v}_i \in \mathbb{R}^n$ for $0 \le i \le n$. The edge lengths are $L_{ij} = |\mathbf{v}_i - \mathbf{v}_j|$. Define M to be the $n \times n$ matrix whose i^{th} row is $\mathbf{v}_i - \mathbf{v}_0$. The hypervolume of the simplex is $V = |\det M|/n!$.

1 Center of Mass

The center of mass of a simplex of homogeneous material is the average of the vertices of the simplex,

$$\mathbf{C} = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{v}_i.$$

The proof is as follows. From calculus, the center of mass for the volume bounded by the simplex S is

$$C_i = \frac{\int_S x_i \, dx_1 \cdots dx_n}{\int_S dx_1 \cdots dx_n}.$$

The numerator of the fraction is a moment about the hyperplane whose normal is in the x_i direction. The denominator of the fraction is the mass of the object. Let $\mathbf{x} = A\mathbf{y} + \mathbf{b}$ be the affine transformation which maps the vertices of the simplex to the $n+1$ points $\mathbf{0}$ and $\mathbf{e}_k$, $1 \le k \le n$, where $\mathbf{e}_k$ has a zero in all components except for a one in the k component. Thus, $\mathbf{b} = \mathbf{v}_0$ and $\mathbf{v}_k = A\mathbf{e}_k + \mathbf{b}$ for $1 \le k \le n$. Let T be the simplex defined by these new points. Change variables in the integrals:

$$\begin{aligned}
C_i &= \frac{\int_S x_i \, dx_1 \cdots dx_n}{\int_S dx_1 \cdots dx_n} \\
&= \frac{\int_T \sum_{j=1}^n a_{ij} y_j + b_i \; |\det(A)| dy_1 \cdots dy_n}{\int_T |\det(A)| dy_1 \cdots dy_n} \\
&= \frac{\int_T \sum_{j=1}^n a_{ij} y_j + b_i \; dy_1 \cdots dy_n}{\int_T dy_1 \cdots dy_n} \\
&= \sum_{j=1}^n a_{ij} \frac{\int_T y_j \; dy_1 \cdots dy_n}{\int_T dy_1 \cdots dy_n} + b_i.
\end{aligned}$$

If $\boldsymbol{\gamma}$ is the center of mass of T, then the above shows that $\mathbf{C} = A\boldsymbol{\gamma} + \mathbf{b}$. That is, the affine transformation of the center of mass is the center of mass of the affinely transformed object.

The symmetry of T requires that the center of mass is $\boldsymbol{\gamma} = (K, \ldots, K)$ for some $K \in (0,1)$. The value $K = I(1)/I(0)$ where

$$\begin{aligned}
I(p-1) &= \int_T x_n^{p-1} \, dx_1 \cdots dx_n \\
&= \int_0^1 \int_0^{1-x_n} \int_0^{1-x_{n-1}-x_n} \cdots \int_0^{1-x_1-x_2-\cdots-x_n} x_n^p \, dx_1 \cdots dx_n \\
&= \int_0^1 x_n^{p-1} \tfrac{1}{(n-1)!} (1-w)^{n-1} \, dx_1 \cdots dx_n \\
&= \beta(p,n), \quad \text{the beta function} \\
&= \tfrac{1}{(n-1)!} \tfrac{\Gamma(p)\Gamma(n)}{\Gamma(p+n)}, \quad \text{Gamma functions} \\
&= \tfrac{1}{(n-1)!} \tfrac{(p-1)!(n-1)!}{(p+n-1)!} \\
&= \tfrac{1}{(p+n-1)\cdots(p+1)p}
\end{aligned}$$

Each integral in the iteration involves integration of a single polynomial term, successive terms increasing in degree by 1. Therefore, $K = [n \cdots 1]/[(n+1) \cdots 2] = 1/(n+1)$. As a result,

$$\boldsymbol{\gamma} = \frac{1}{n+1}\left(\sum_{i=1}^{n} \mathbf{e}_i + \mathbf{0}\right)$$

which is the average of the $n+1$ vertices of T. Moreover,

$$\begin{aligned}
\mathbf{C} &= A\boldsymbol{\gamma} + \mathbf{b} \\
&= A\left(\tfrac{1}{n+1}\textstyle\sum_{i=1}^{n} \mathbf{e}_i\right) + \mathbf{b} \\
&= \frac{A\sum_{i=1}^{n} \mathbf{e}_i + (n+1)\mathbf{b}}{n+1} \\
&= \frac{\sum_{i=1}^{n} (A\mathbf{e}_i + \mathbf{b}) + \mathbf{b}}{n+1} \\
&= \frac{\sum_{i=1}^{n} \mathbf{v}_i + \mathbf{v}_0}{n+1},
\end{aligned}$$

so the center of mass $\mathbf{C}$ of simplex S is the average of the vertices of S.

2 Circumscribed Hypersphere

A circumscribing hypersphere for the simplex is that hypersphere passing through all the vertices of the simplex. The center of this hypersphere, $\mathbf{C}$, is equidistant from the vertices, say of distance r. The constraints are

$$|\mathbf{C} - \mathbf{v}_i| = r, \;\; 0 \le i \le n.$$

Squaring the equations, expanding the dot products, and subtracting the equation for $i = 0$ yields

$$(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{C} - \mathbf{v}_0) = \frac{1}{2}|\mathbf{v}_i - \mathbf{v}_0|^2 = \frac{1}{2}L_{i0}^2, \;\; 1 \le i \le n.$$

This is a system of linear equations in the components of $\mathbf{C}$. Let M be the $n \times n$ matrix defined earlier. Let $\mathbf{b}$ be the $n \times 1$ vector whose i^{th} row is $L_{i0}^2/2$. The equation defining the center is $M(\mathbf{C} - \mathbf{v}_0) = \mathbf{b}$ and has solution

$$\mathbf{C} = \mathbf{v}_0 + M^{-1}\mathbf{b}.$$

The radius of the circumscribed hypersphere is

$$r = |\mathbf{C} - \mathbf{v}_0| = \left|M^{-1}\mathbf{b}\right|.$$

For $n = 2$ the simplex is a triangle. The area is

$$A = |(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)|/2.$$

The center (x, y) and radius r are

$$\begin{aligned}
x &= x_0 + \tfrac{1}{4A}(+(y_2 - y_0)L_{10}^2 - (y_1 - y_0)L_{20}^2) \\
y &= y_0 + \tfrac{1}{4A}(-(x_2 - x_0)L_{10}^2 + (x_1 - x_0)L_{20}^2) \\
r &= \sqrt{(x - x_0)^2 + (y - y_0)^2}.
\end{aligned}$$

For $n = 3$ the simplex is a tetrahedron. Define $X_i = x_i - x_0$, $Y_i = y_i - y_0$, and $Z_i = z_i - z_0$ for $1 \le i \le 3$. The volume is

$$V = \frac{1}{6}\left|\det\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix}\right|.$$

The center (x, y, z) and radius r are

$$\begin{aligned} x &= x_0 + \tfrac{1}{12V}\left(+(Y_2Z_3 - Y_3Z_2)L_{10}^2 - (Y_1Z_3 - Y_3Z_1)L_{20}^2 + (Y_1Z_2 - Y_2Z_1)L_{30}^2\right) \\ y &= y_0 + \tfrac{1}{12V}\left(-(X_2Z_3 - X_3Z_2)L_{10}^2 + (X_1Z_3 - X_3Z_1)L_{20}^2 - (X_1Z_2 - X_2Z_1)L_{30}^2\right) \\ z &= z_0 + \tfrac{1}{12V}\left(+(X_2Y_3 - X_3Y_2)L_{10}^2 - (X_1Y_3 - X_3Y_1)L_{20}^2 + (X_1Y_2 - X_2Y_1)L_{30}^2\right) \\ r &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \end{aligned}$$

3 Inscribed Hypersphere

An inscribing hypersphere for the simplex is that hypersphere which is contained entirely within the simplex and is tangent to all faces of the simple. The center of the hypersphere, $\mathbf{C}$, is equidistant from the faces, say of distance r. The constraints are

$$\mathbf{N}_i \cdot (\mathbf{C} - \mathbf{v}_i) = r, \quad 0 \le i \le n,$$

where $\mathbf{N}_i$ is the inner unit normal to the hyperface determined by the vertices $\mathbf{v}_{i \bmod n}$, $\mathbf{v}_{(i+1) \bmod n}$, $\cdots$, $\mathbf{v}_{(i+n-1) \bmod n}$. This is a linear system of $n + 1$ equations in the $n + 1$ unknowns $(\mathbf{C}, r)$. The system can be written as

$$(\mathbf{N}_i, -1) \cdot (\mathbf{C}, r) = \mathbf{N}_i \cdot \mathbf{v}_i.$$

Define the $(n + 1) \times (n + 1)$ matrix M to be that matrix whose i^{th} row is the vector $(\mathbf{N}_i, -1)$. Define the $(n+1) \times 1$ vector $\mathbf{b}$ to be that vector whose i^{th} row is the scalar $\mathbf{N}_i \cdot \mathbf{v}_i$. The linear system is then $M\mathbf{C} = \mathbf{b}$ and has solution

$$(\mathbf{C}, r) = M^{-1}\mathbf{b}.$$

The radius of the inscribed hypersphere is

$$r = \mathbf{N}_0 \cdot (\mathbf{C} - \mathbf{v}_0).$$

For $n = 2$ the normals are

$$\begin{aligned} \mathbf{N}_0 &= \tfrac{(-(y_1 - y_0), +(x_1 - x_0))}{L_{01}} \\ \mathbf{N}_1 &= \tfrac{(-(y_2 - y_1), +(x_2 - x_1))}{L_{12}} \\ \mathbf{N}_2 &= \tfrac{(-(y_0 - y_2), +(x_0 - x_2))}{L_{20}} \end{aligned}$$

The system of equations is

$$\begin{bmatrix} -(y_1 - y_0) & (x_1 - x_0) & -L_{01} \\ -(y_2 - y_1) & (x_2 - x_1) & -L_{12} \\ -(y_0 - y_2) & (x_0 - x_2) & -L_{20} \end{bmatrix} \begin{bmatrix} x \\ y \\ r \end{bmatrix} = \begin{bmatrix} x_1y_0 - x_0y_1 \\ x_2y_1 - x_1y_2 \\ x_0y_2 - x_2y_0 \end{bmatrix}.$$

Applying a symbolic inversion yields solution

$$\begin{aligned} x &= \frac{x_0 L_{12}+x_1 L_{20}+x_2 L_{01}}{L_{01}+L_{12}+L_{20}} \\ y &= \frac{y_0 L_{12}+y_1 L_{20}+y_2 L_{01}}{L_{01}+L_{12}+L_{20}} \\ r &= \frac{x_0 y_1-x_1 y_0+x_1 y_2-x_2 y_1+x_2 y_0-x_0 y_2}{L_{01}+L_{12}+L_{20}} \end{aligned}$$

For $n = 3$ the normals are

$$\begin{aligned} \mathbf{N}_0 &= \frac{1}{L_0}(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0) \\ \mathbf{N}_1 &= \frac{1}{L_1}(\mathbf{v}_3 - \mathbf{v}_0) \times (\mathbf{v}_1 - \mathbf{v}_0) \\ \mathbf{N}_2 &= \frac{1}{L_2}(\mathbf{v}_2 - \mathbf{v}_0) \times (\mathbf{v}_3 - \mathbf{v}_0) \\ \mathbf{N}_3 &= \frac{1}{L_3}(\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_2 - \mathbf{v}_1) \end{aligned}$$

where the denominators L_k are the lengths of the corresponding vectors in the numerators. The 4×4 system of equations can be solved symbolically to yield

$$\begin{aligned} x &= \frac{x_0 L_3+x_1 L_2+x_2 L_1+x_3 L_0}{L_0+L_1+L_2+L_3} \\ y &= \frac{y_0 L_3+y_1 L_2+y_2 L_1+y_3 L_0}{L_0+L_1+L_2+L_3} \\ z &= \frac{z_0 L_3+z_1 L_2+z_2 L_1+z_3 L_0}{L_0+L_1+L_2+L_3} \\ r &= \frac{\gamma}{L_0+L_1+L_2+L_3} \end{aligned}$$

where γ is the absolute value of the sum of the components of the generalized cross product

$$\begin{bmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{bmatrix}$$