Reconstructing an Ellipsoid from its Perspective Projection onto a Plane

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Created: May 5, 2007
Last Modified: March 1, 2008

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1 Introduction

An ellipsoid in standard form is

$$(\mathbf{X}-\mathbf{C})^{\mathrm{T}}A(\mathbf{X}-\mathbf{C}) = 1 \tag{1}$$

where $\mathbf{C}$ is the center of the ellipsoid, where A is positive definite (all positive eigenvalues), and where $\mathbf{X}$ is any point on the ellipsoid.

Given an eyepoint $\mathbf{E}$ and a projection plane $\mathbf{N}\cdot\mathbf{Y} = d$, where $\mathbf{N}$ is a unit-length vector, d is a constant, $\mathbf{Y}$ is any point on the plane, and $\mathbf{E}$ is not on the plane, we may project the ellipsoid to an ellipse on the plane. As a 3D quantity, the ellipse is

$$\mathbf{Y} = \mathbf{K} + u\mathbf{U} + v\mathbf{V} \tag{2}$$

where $\mathbf{K}$ is the center of the ellipse, where $\{\mathbf{U},\mathbf{V},\mathbf{N}\}$ is an orthonormal set (unit-length vectors that are mutually orthogonal), and where $(u/a)^2+(v/b)^2 = 1$. The values a and b are the major and minor axis half-lengths. The document Perspective Projection of an Ellipsoid provides the algebraic details of computing $\mathbf{K}$, $\mathbf{U}$, $\mathbf{V}$, a, and b from the known quantities A and $\mathbf{C}$.

This document describes how to reconstruct an ellipsoid from the ellipse. That is, if $\mathbf{E}$, $\mathbf{N}$, $\mathbf{K}$, $\mathbf{U}$, $\mathbf{V}$, a, and b are known, we wish to compute a matrix A and $\mathbf{C}$ for an ellipsoid whose perspective projection is the ellipse. As it turns out, there are infinitely many solutions. You can see this by considering an eyepoint at $(0,0,1)$, a projection plane of $z = 0$, and a circle centered at $(0,0,0)$ of radius r. There are infinitely many ellipsoids of the form $(x^2+y^2)/\alpha^2+(z-s)^2/\beta^2 = 1$ for $s < 1$ and whose perspective projections are the specified circle. At the end of this document, I will use this example to reconstruct ellipsoids that project to the circle.

2 Determining the Elliptical Cone Bounding the Ellipsoid

Let us first start with an ellipsoid and compute the elliptical cone that tightly bounds the ellipsoid. Consider a ray $\mathbf{L}(t) = \mathbf{E}+t\mathbf{D}$, where $t > 0$ and $\mathbf{D}$ is a unit-length vector. The intersection of the ray and ellipsoid is determined by

$$1 = (\mathbf{L}(t)-\mathbf{C})^{\mathrm{T}}A(\mathbf{L}(t)-\mathbf{C}) \tag{3}$$

or equivalently by

$$t^2\mathbf{D}^{\mathrm{T}}A\mathbf{D} + 2t\boldsymbol{\Delta}^{\mathrm{T}}A\mathbf{D} + \boldsymbol{\Delta}^{\mathrm{T}}A\boldsymbol{\Delta} - 1 = 0 \tag{4}$$

where $\boldsymbol{\Delta} = \mathbf{E}-\mathbf{C}$. This is a quadratic equation in t. If this equation has two distinct real-valued roots, the ray intersects the ellipsoid twice. If this equation has no real-valued roots, the ray does not intersect the ellipsoid. When the equation has a single real-valued root, the ray is tangent to the ellipsoid. The condition for having a single real-valued root is that the discriminant is zero:

$$(\boldsymbol{\Delta}^{\mathrm{T}}A\mathbf{D})^2 - (\mathbf{D}^{\mathrm{T}}A\mathbf{D})(\boldsymbol{\Delta}^{\mathrm{T}}A\boldsymbol{\Delta}-1) = 0 \tag{5}$$

This equation factors into

$$\mathbf{D}^{\mathrm{T}}\left(A^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}}A - (\boldsymbol{\Delta}^{\mathrm{T}}A\boldsymbol{\Delta}-1)A\right)\mathbf{D} = 0 \tag{6}$$

The unit-length directions are determined by the tangent points, $\mathbf{D} = (\mathbf{X}-\mathbf{E})/|\mathbf{X}-\mathbf{E}|$. We may substitute this into the equation and multiply through by the denominator to obtain

$$(\mathbf{X}-\mathbf{E})^{\mathrm{T}}B(\mathbf{X}-\mathbf{E}) = 0 \tag{7}$$

where

$$B = A^{\mathrm{T}}\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}}A - (\boldsymbol{\Delta}^{\mathrm{T}}A\boldsymbol{\Delta} - 1)A \tag{8}$$

Equation (7) describes an elliptical cone whose vertex is at $\mathbf{E}$. By the geometry, the matrix B is nonsingular and has two eigenvalues of the same sign and another eigenvalue of the opposite sign. If the two eigenvalues of the same sign are equal, then the cone is circular.

3 Reconstructing the Elliptical Cone Bounding the Ellipsoid

For any point $\mathbf{Y}$ on the ellipse, the corresponding ellipsoid point $\mathbf{X}$ must be on the line segment from the eyepoint to the ellipse point, which implies that $\mathbf{X} - \mathbf{E}$ and $\mathbf{Y} - \mathbf{E}$ are parallel vectors. As such, $\mathbf{X} - \mathbf{E} = t(\mathbf{Y} - \mathbf{E})$ for some $t > 0$. Dotting the equation with $\mathbf{N}$ and rearranging terms produces

$$\frac{\mathbf{X} - \mathbf{E}}{\mathbf{N} \cdot (\mathbf{X} - \mathbf{E})} = \frac{\mathbf{Y} - \mathbf{E}}{\mathbf{N} \cdot (\mathbf{Y} - \mathbf{E})} \tag{9}$$

Substituting Equation (2) into Equation (9) and rearranging some terms leads to

$$r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E}) = u\mathbf{U} + v\mathbf{V} \tag{10}$$

where $r = \mathbf{N} \cdot (\mathbf{K} - \mathbf{E})/\mathbf{N} \cdot (\mathbf{X} - \mathbf{E})$. I have used the fact that $\mathbf{N}$ is orthogonal to both $\mathbf{U}$ and $\mathbf{V}$. Dot Equation (10) with $\mathbf{U}$ and then with $\mathbf{V}$ to obtain

$$u = \mathbf{U} \cdot (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})) \;\; v = \mathbf{V} \cdot (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})) \tag{11}$$

Substituting these into $(u/a)^2 + (v/b)^2 = 1$,

$$\begin{aligned} 1 &= \left(\frac{\mathbf{U}\cdot(r(\mathbf{X}-\mathbf{E})-(\mathbf{K}-\mathbf{E}))}{a}\right)^2 + \left(\frac{\mathbf{V}\cdot(r(\mathbf{X}-\mathbf{E})-(\mathbf{K}-\mathbf{E}))}{b}\right)^2 \\ &= (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E}))^{\mathrm{T}} \left(\frac{\mathbf{U}\mathbf{U}^{\mathrm{T}}}{a^2} + \frac{\mathbf{V}\mathbf{V}^{\mathrm{T}}}{b^2}\right) (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})) \\ &= (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E}))^{\mathrm{T}} M (r(\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})) \end{aligned} \tag{12}$$

where the last equality defines the matrix M. Multiply Equation (12) by $1/r^2$ to obtain

$$\frac{1}{r^2} = ((\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})/r)^{\mathrm{T}} M ((\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})/r) \tag{13}$$

Now observe that

$$\begin{aligned} (\mathbf{X} - \mathbf{E}) - (\mathbf{K} - \mathbf{E})/r &= (\mathbf{X} - \mathbf{E}) - \frac{(\mathbf{K}-\mathbf{E})}{\mathbf{N}\cdot(\mathbf{K}-\mathbf{E})}\, \mathbf{N} \cdot (\mathbf{X} - \mathbf{E}) \\ &= \left(I - \frac{(\mathbf{K}-\mathbf{E})}{\mathbf{N}\cdot(\mathbf{K}-\mathbf{E})}\mathbf{N}^{\mathrm{T}}\right)(\mathbf{X} - \mathbf{E}) \\ &= P(\mathbf{X} - \mathbf{E}) \end{aligned} \tag{14}$$

where I is the identity matrix and where the last equality defines the matrix P. Also observe that

$$\frac{1}{r^2} = \frac{(\mathbf{N} \cdot (\mathbf{X} - \mathbf{E}))^2}{(\mathbf{N} \cdot (\mathbf{K} - \mathbf{E}))^2} = \frac{(\mathbf{X} - \mathbf{E})^{\mathrm{T}}\mathbf{N}\mathbf{N}^{\mathrm{T}}(\mathbf{X} - \mathbf{E})}{(\mathbf{N} \cdot (\mathbf{K} - \mathbf{E}))^2} = (\mathbf{X} - \mathbf{E})^{\mathrm{T}} Q (\mathbf{X} - \mathbf{E}) \tag{15}$$

where the last equality defines the matrix Q.

Substitute Equations (14) and (15) into Equation (13) to obtain

$$0 = (\mathbf{X} - \mathbf{E})^{\mathrm{T}} \left(P^{\mathrm{T}} M P - Q\right) (\mathbf{X} - \mathbf{E}) = (\mathbf{X} - \mathbf{E})^{\mathrm{T}} B' (\mathbf{X} - \mathbf{E}) \tag{16}$$

where $B' = P^{\mathrm{T}} M P - Q$. Compare this to Equation (8), since the current equation is itself that of an elliptical cone. In fact, using any nonzero multiple of B', say, $\sigma B'$ for $\sigma \neq 0$, in the cone equation does not change the solutions. We will need this extra degree of freedom in reconstructing A and $\mathbf{C}$.

4 Reconstruction of an Ellipsoid

The problem now reduces to choosing a center point $\mathbf{C}$ and a positive definite matrix A that define the ellipsoid and for which

$$A^{\mathrm{T}} \mathbf{\Delta} \mathbf{\Delta}^{\mathrm{T}} A - (\mathbf{\Delta}^{\mathrm{T}} A \mathbf{\Delta} - 1) A = B \tag{17}$$

where $B = \sigma B'$ and $\mathbf{\Delta} = \mathbf{E} - \mathbf{C}$. Although you might be tempted to choose $\mathbf{C}$ on the ray from the eyepoint $\mathbf{E}$ through the ellipse center $\mathbf{K}$, it is generally not the case that $\mathbf{C}$ projects to $\mathbf{K}$. An example to illustrate: Project the sphere $x^2 + y^2 + z^2 = 1$ onto the plane $z = 0$ using the eyepoint $\mathbf{E} = (-1, 0, 2)$. The projection of the sphere center $\mathbf{C} = (0, 0, 0)$ onto $z = 0$ is the point $(0, 0, 0)$. However, the projection of the sphere is an ellipse whose center is $\mathbf{K} = (1/3, 0, 0)$.

Multiply Equation (17) on the right by $\mathbf{\Delta}$ to obtain $A\mathbf{\Delta} = B\mathbf{\Delta}$. Replacing this back into the equation, we obtain

$$B^{\mathrm{T}} \mathbf{\Delta} \mathbf{\Delta}^{\mathrm{T}} B - (\mathbf{\Delta}^{\mathrm{T}} B \mathbf{\Delta} - 1) A = B \tag{18}$$

If A is a positive definite matrix and $\mathbf{C}$ is a vector in front of the eyepoint, both which define an ellipsoid that projects onto the ellipse, they must cause Equation (18) to be true. However, Equation (18) might have extraneous solutions that do not lead to an ellipsoid. For example, it is possible that symmetric matrices A are solutions but those matrices are not positive definite.

If you were to choose a candidate positive definite matrix A, then you can compute solutions $\mathbf{\Delta}$ directly from Equation (18). If you were to choose a candidate center $\mathbf{C}$, then $\mathbf{\Delta}$ is a known quantity and you may formally solve Equation (18) for A,

$$A = \frac{B - B^{\mathrm{T}} \mathbf{\Delta} \mathbf{\Delta} B}{1 - \mathbf{\Delta}^{\mathrm{T}} B \mathbf{\Delta}} = \frac{\sigma B' - \sigma^2 (B')^{\mathrm{T}} \mathbf{\Delta} \mathbf{\Delta} B'}{1 - \sigma \mathbf{\Delta}^{\mathrm{T}} B' \mathbf{\Delta}} \tag{19}$$

However, observe that the computed A might not be positive definite, in which case your specified center is not possible *for the choice you made for σ when computing $B = \sigma B'$*. Varying the value of σ is necessary to force a positive definite solution A.

Example. Consider the simple case where the eyepoint is $\mathbf{E} = (0, 0, 1)$ and the projection plane is $z = 0$, in which case $\mathbf{N} = (0, 0, 1)$. The projection is chosen to be a circle with center $\mathbf{K} = (0, 0, 0)$ and radius 1, in which case $a = b = 1$, and we may choose $\mathbf{U} = (1, 0, 0)$ and $\mathbf{V} = (0, 1, 0)$.

Choose a center $\mathbf{C} = (0, 0, s)$ for $s < 1$. You will find that $\mathbf{\Delta} = (0, 0, s - 1)$, $P = M = \mathrm{Diag}(1, 1, 0)$, $Q = \mathrm{Diag}(0, 0, 1)$, $B = \mathrm{Diag}(1, 1, -1)$, and finally

$$A = \mathrm{Diag}\left(\frac{\sigma}{1 + \sigma(s-1)^2}, \frac{\sigma}{1 + \sigma(s-1)^2}, -\sigma\right)$$

For A to be positive definite, we need $\sigma < 0$ for the last diagonal entry to be positive. For the first two diagonal entries to be positive, we need $\sigma < -1/(s-1)^2$. The reconstructed ellipsoid is

- a sphere when $\sigma = -2/(s-1)^2$,
- a prolate spheroid (rotation of ellipse about major axis) when $\sigma < -2/(s-1)^2$, or
- an oblate spheroid (rotation of ellipse about minor axis) when $-2/(s-1)^2 < \sigma < -1/(s-1)^2$.

This example implies that, generally, you have multiple degrees of freedom. In the example at hand, you may choose the location of the ellipsoid center along the z-axis (choose any $s < 1$) and you may choose the shape of the ellipsoid (choose any $\sigma < -1/(s-1)^2$).

5 A Summary of the Algorithm

The following are specified quantities:

- The eyepoint $\mathbf{E}$.
- A unit-length normal $\mathbf{N}$ for the projection plane.
- The ellipse center $\mathbf{K}$, which is a point on the projection plane.
- The ellipse axes $\mathbf{U}$ and $\mathbf{V}$, both unit length and orthogonal to each other, and both orthogonal to $\mathbf{N}$.
- The ellipse axis half-lengths a, corresponding to axis $\mathbf{U}$, and b, corresponding to axis $\mathbf{V}$.

The following are computed from the specified quantities:

- The matrix $M = \mathbf{U}\mathbf{U}^{\mathrm{T}}/a^2 + \mathbf{V}\mathbf{V}^{\mathrm{T}}/b^2$.
- The vector $\mathbf{W} = \mathbf{N}/(\mathbf{N} \cdot (\mathbf{K} - \mathbf{E}))$.
- The matrix $P = I - (\mathbf{K} - \mathbf{E})\mathbf{W}^{\mathrm{T}}$.
- The matrix $Q = \mathbf{W}\mathbf{W}^{\mathrm{T}}$.
- The matrix $B' = P^{\mathrm{T}}MP - Q$.

When you choose a center $\mathbf{C}$ and scale σ, compute $\mathbf{\Delta} = \mathbf{E} - \mathbf{C}$ and $B = \sigma B'$, and then compute $A = (B - B^{\mathrm{T}}\mathbf{\Delta}\mathbf{\Delta}B)/(1 - \mathbf{\Delta}^{\mathrm{T}}B\mathbf{\Delta})$. However, A might not be positive definite, in which case you must choose a different σ. To check if A is positive definite, you can compute an eigendecomposition $A = RDR^{\mathrm{T}}$, where R is an orthogonal matrix and where D is a diagonal matrix. You are good to go when the diagonal entries of D are all positive. Alternatively, you can compute the coefficients of the characteristic polynomial $p(t) = \det(tI - A)$ and use the Routh-Hurwitz Criterion for determining if all the roots are positive. This is a simple algebraic test and does not require actually computing the roots as an eigendecomposition does. Equivalent to this is to compute a Sturm sequence of polynomials for $p(t)$ and count the number of roots on the interval $(0, \infty)$. If the answer is three, you have a positive definite matrix A.