# Euler Angle Formulas

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## Contents

# 1 Introduction

Rotations about the coordinate axes are easy to define and work with. Rotation about the $x$-axis by angle $\theta$ is

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $x = 0$. The observer is assumed to be positioned on the side of the plane with $x > 0$ and looking at the origin. Rotation about the $y$-axis by angle $\theta$ is

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $y = 0$. The observer is assumed to be positioned on the side of the plane with $y > 0$ and looking at the origin. Rotation about the $z$-axis by angle $\theta$ is

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $z = 0$. The observer is assumed to be positioned on the side of the plane with $z > 0$ and looking at the origin. Rotation by an angle $\theta$ about an arbitrary axis containing the origin and having unit length direction $\mathbf{U} = (U_x, U_y, U_z)$ is given by

$$R_{\mathbf{U}}(\theta) = I + (\sin\theta)S + (1 - \cos\theta)S^2$$

where $I$ is the identity matrix,

$$S = \begin{bmatrix} 0 & -U_z & U_y \\ U_z & 0 & -U_x \\ -U_y & U_x & 0 \end{bmatrix}$$

and $\theta > 0$ indicates a counterclockwise rotation in the plane $\mathbf{U} \cdot (x, y, z) = 0$. The observer is assumed to be positioned on the side of the plane to which $\mathbf{U}$ points and is looking at the origin.

# 2 Factor as a Product of Three Rotation Matrices

A common problem is to factor a rotation matrix as a product of rotations about the coordinate axes. The form of the factorization depends on the needs of the application and what ordering is specified. For example, one might want to factor a rotation as $R = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$ for some angles $\theta_x$, $\theta_y$, and $\theta_z$. The ordering is $xyz$. Five other possibilities are $xzy$, $yxz$, $yzx$, $zxy$, and $zyx$. It is also possible to factor as $R = R_x(\theta_{x_0})R_y(\theta_y)R_x(\theta_{x_1})$, the ordering referred to as $xyx$. Five other possibilites are $xzx$, $yxy$, $yzy$, $zxz$, and $zyz$. These are also discussed here. The following discussion uses the notation $c_a = \cos(\theta_a)$ and $s_a = \sin(\theta_a)$ for $a = x, y, z$.

## 2.1 Factor as $R_xR_yR_z$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_yc_z & -c_ys_z & s_y \\ c_zs_xs_y + c_xs_z & c_xc_z - s_xs_ys_z & -c_ys_x \\ -c_xc_zs_y + s_xs_z & c_zs_x + c_xs_ys_z & c_xc_y \end{bmatrix}$$

The simplest term to work with is $s_y = r_{02}$, so $\theta_y = \text{asin}(r_{02})$. There are three cases to consider.

Case 1: If $\theta_y \in (-\pi/2, \pi/2)$, then $c_y \neq 0$ and $c_y(s_x, c_x) = (-r_{12}, r_{22})$, in which case $\theta_x = \text{atan2}(-r_{12}, r_{22})$, and $c_y(s_z, c_z) = (-r_{01}, r_{00})$, in which case $\theta_z = \text{atan2}(-r_{01}, r_{00})$. In summary,

$$\theta_y = \text{asin}(r_{02}), \;\; \theta_x = \text{atan2}(-r_{12}, r_{22}). \;\; \theta_z = \text{atan2}(-r_{01}, r_{00}) \tag{1}$$

Case 2: If $\theta_y = \pi/2$, then $s_y = 1$ and $c_y = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{11} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} c_zs_x + c_xs_z & c_xc_z - s_xs_z \\ -c_xc_z + s_xs_z & c_zs_x + c_xs_z \end{bmatrix} = \begin{bmatrix} \sin(\theta_z + \theta_x) & \cos(\theta_z + \theta_x) \\ -\cos(\theta_z + \theta_x) & \sin(\theta_z + \theta_x) \end{bmatrix}.$$

Therefore, $\theta_z + \theta_x = \text{atan2}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = \pi/2, \;\; \theta_z + \theta_x = \text{atan2}(r_{10}, r_{11}) \tag{2}$$

Case 3: If $\theta_y = -\pi/2$, then $s_y = -1$ and $c_y = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{11} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} -c_zs_x + c_xs_z & c_xc_z + s_xs_z \\ c_xc_z + s_xs_z & c_zs_x - c_xs_z \end{bmatrix} = \begin{bmatrix} \sin(\theta_z - \theta_x) & \cos(\theta_z - \theta_x) \\ \cos(\theta_z - \theta_x) & -\sin(\theta_z - \theta_x) \end{bmatrix}.$$

Therefore, $\theta_z - \theta_x = \text{atan2}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = -\pi/2, \;\; \theta_z - \theta_x = \text{atan2}(r_{10}, r_{11}) \tag{3}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{02}$ is tested for the three cases.

```
if (r02 < +1)
{
    if (r02 > -1)
    {
        thetaY = asin(r02);
        thetaX = atan2(-r12,r22);
        thetaZ = atan2(-r01,r00);
    }
    else  // r02 = -1
```

```
    {
        // Not a unique solution:  thetaZ - thetaX = atan2(r10,r11)
        thetaY = -PI/2;
        thetaX = -atan2(r10,r11);
        thetaZ = 0;
    }
}
else  // r02 = +1
{
    // Not a unique solution:  thetaZ + thetaX = atan2(r10,r11)
    thetaY = +PI/2;
    thetaX = atan2(r10,r11);
    thetaZ = 0;
}
```

## 2.2 Factor as $R_xR_zR_y$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_x(\theta_x)R_z(\theta_z)R_y(\theta_y)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_yc_z & -s_z & c_zs_y \\ s_xs_y + c_xc_ys_z & c_xc_z & -c_ys_x + c_xs_ys_z \\ -c_xs_y + c_ys_xs_z & c_zs_x & c_xc_y + s_xs_ys_z \end{bmatrix}$$

The simplest term to work with is $-s_z = r_{01}$, so $\theta_z = \text{asin}(-r_{01})$. There are three cases to consider.

Case 1: If $\theta_z \in (-\pi/2, \pi/2)$, then $c_z \neq 0$ and $c_z(s_y, c_y) = (r_{02}, r_{00})$, in which case $\theta_y = \text{atan2}(r_{02}, r_{00})$, and $c_z(s_x, c_x) = (r_{21}, r_{11})$, in which case $\theta_x = \text{atan2}(r_{21}, r_{11})$. In summary,

$$\theta_z = \text{asin}(-r_{01}), \;\; \theta_x = \text{atan2}(r_{21}, r_{11}), \;\; \theta_y = \text{atan2}(r_{02}, r_{00}) \tag{4}$$

Case 2: If $\theta_z = \pi/2$, then $s_z = 1$ and $c_z = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{12} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} s_xs_y + c_xc_y & -c_ys_x + c_xs_y \\ -c_xs_y + c_ys_x & c_xc_y + s_xs_y \end{bmatrix} = \begin{bmatrix} \cos(\theta_y - \theta_x) & \sin(\theta_y - \theta_x) \\ -\sin(\theta_y - \theta_x) & \cos(\theta_y - \theta_x) \end{bmatrix}$$

Therefore, $\theta_y - \theta_x = \text{atan2}(-r_{20}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = +\pi/2, \;\; \theta_y - \theta_x = \text{atan2}(-r_{20}, r_{22}) \tag{5}$$

Case 3: If $\theta_z = -\pi/2$, then $s_z = -1$ and $c_z = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{12} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} s_xs_y - c_xc_y & -c_ys_x - c_xs_y \\ -c_xs_y - c_ys_x & c_xc_y - s_xs_y \end{bmatrix} = \begin{bmatrix} -\cos(\theta_y + \theta_x) & -\sin(\theta_y + \theta_x) \\ -\sin(\theta_y + \theta_x) & \cos(\theta_y + \theta_x) \end{bmatrix}$$

Therefore, $\theta_y + \theta_x = \text{atan2}(-r_{20}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = -\pi/2, \;\; \theta_y + \theta_x = \text{atan2}(-r_{20}, r_{22}) \tag{6}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{01}$ is tested for the three cases.

```
if (r01 < +1)
{
    if (r01 > -1)
    {
        thetaZ = asin(-r01);
        thetaX = atan2(r21,r11);
        thetaY = atan2(r02,r00);
    }
    else // r01 = -1
    {
        // Not a unique solution:  thetaY - thetaX = atan2(-r20,r22)
        theta_z = +pi/2;
        theta_x = atan2(-r20,r22);
        theta_y = 0;
    }
}
else // r01 = +1
{
    // Not a unique solution:  thetaY + thetaX = atan2(-r20,r22)
    theta_z = -pi/2;
    theta_x = atan2(-r20,r22);
    theta_y = 0;
}
```

### 2.3 Factor as $R_yR_xR_z$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_y(\theta_y)R_x(\theta_x)R_z(\theta_z)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_yc_z + s_xs_ys_z & c_zs_xs_y - c_ys_z & c_xs_y \\ c_xs_z & c_xc_z & -s_x \\ -c_zs_y + c_ys_xs_z & c_yc_zs_x + s_ys_z & c_xc_y \end{bmatrix}$$

The simplest term to work with is $-s_x = r_{12}$, so $\theta_x = \text{asin}(-r_{12})$. There are three cases to consider.

Case 1: If $\theta_x \in (-\pi/2, \pi/2)$, then $c_x \neq 0$ and $c_x(s_y, c_y) = (r_{02}, r_{22})$, in which case $\theta_y = \text{atan2}(r_{02}, r_{22})$, and $c_x(s_z, c_z) = (r_{10}, r_{11})$, in which case $\theta_z = \text{atan2}(r_{10}, r_{11})$. In summary,

$$\theta_x = \text{asin}(-r_{12}), \;\; \theta_y = \text{atan2}(r_{02}, r_{22}), \;\; \theta_z = \text{atan2}(r_{10}, r_{11}) \tag{7}$$

Case 2: If $\theta_x = \pi/2$, then $s_x = 1$ and $c_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} c_yc_z + s_ys_z & c_zs_y - c_ys_z \\ -c_zs_y + c_ys_z & c_yc_z + s_ys_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_z - \theta_y) & -\sin(\theta_z - \theta_y) \\ \sin(\theta_z - \theta_y) & \cos(\theta_z - \theta_y) \end{bmatrix}$$

Therefore, $\theta_z - \theta_y = \text{atan2}(-r_{01}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = +\pi/2, \;\; \theta_z - \theta_y = \text{atan2}(-r_{01}, r_{00}) \tag{8}$$

CASE 3: If $\theta_x = -\pi/2$, then $s_x = -1$ and $c_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} c_y c_z - s_y s_z & -c_z s_y - c_y s_z \\ -c_z s_y - c_y s_z & -c_y c_z + s_y s_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_z + \theta_y) & -\sin(\theta_z + \theta_y) \\ -\sin(\theta_z + \theta_y) & -\cos(\theta_z + \theta_y) \end{bmatrix}$$

Therefore, $\theta_z + \theta_y = \text{atan2}(-r_{01}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = -\pi/2, \;\; \theta_z + \theta_y = \text{atan2}(-r_{01}, r_{00}) \tag{9}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{12}$ is tested for the three cases.

```
if (r12 < +1)
{
    if (r12 > -1)
    {
        thetaX = asin(-r12);
        thetaY = atan2(r02,r22);
        thetaZ = atan2(r10,r11);
    }
    else // r12 = -1
    {
        // Not a unique solution:  thetaZ - thetaY = atan2(-r01,r00)
        thetaX = +pi/2;
        thetaY = -atan2(-r01,r00);
        thetaZ = 0;
    }
}
else // r12 = +1
{
    // Not a unique solution:  thetaZ + thetaY = atan2(-r01,r00)
    thetaX = -pi/2;
    thetaY = atan2(-r01,r00);
    thetaZ = 0;
}
```

## 2.4 Factor as $R_y R_z R_x$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_y(\theta_y)R_z(\theta_z)R_x(\theta_x)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z & s_x s_y - c_x c_y s_z & c_x s_y + c_y s_x s_z \\ s_z & c_x c_z & -c_z s_x \\ -c_z s_y & c_y s_x + c_x s_y s_z & c_x c_y - s_x s_y s_z \end{bmatrix}$$

The simplest term to work with is $s_z = r_{10}$, so $\theta_z = \text{asin}(r_{10})$. There are three cases to consider.

CASE 1: If $\theta_z \in (-\pi/2, \pi/2)$, then $c_z \neq 0$ and $c_z(s_x, c_x) = (-r_{12}, r_{11})$, in which case $\theta_x = \text{atan2}(-r_{12}, r_{11})$, and $c_z(s_y, c_y) = (-r_{20}, r_{00})$, in which case $\theta_y = \text{atan2}(-r_{20}, r_{00})$. In summary,

$$\theta_z = \text{asin}(r_{10}), \;\; \theta_y = \text{atan2}(-r_{20}, r_{00}), \;\; \theta_x = \text{atan2}(-r_{12}, r_{11}) \tag{10}$$

CASE 2: If $\theta_z = \pi/2$, then $s_z = 1$ and $c_z = 0$. In this case,

$$\begin{bmatrix} r_{01} & r_{02} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} s_x s_y - c_x c_y & c_x s_y + c_y s_x \\ c_y s_x + c_x s_y & c_x c_y - s_x s_y \end{bmatrix} = \begin{bmatrix} -\cos(\theta_x + \theta_y) & \sin(\theta_x + \theta_y) \\ \sin(\theta_x + \theta_y) & \cos(\theta_x + \theta_y) \end{bmatrix}$$

Therefore, $\theta_x + \theta_y = \text{atan2}(r_{21}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = +\pi/2, \;\; \theta_x + \theta_y = \text{atan2}(r_{21}, r_{22}) \tag{11}$$

CASE 3: If $\theta_z = -\pi/2$, then $s_z = -1$ and $c_z = 0$. In this case,

$$\begin{bmatrix} r_{01} & r_{02} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} s_x s_y + c_x c_y & c_x s_y - c_y s_x \\ c_y s_x - c_x s_y & c_x c_y + s_x s_y \end{bmatrix} = \begin{bmatrix} \cos(\theta_x - \theta_y) & -\sin(\theta_x - \theta_y) \\ \sin(\theta_x - \theta_y) & \cos(\theta_x - \theta_y) \end{bmatrix}$$

Therefore, $\theta_x - \theta_y = \text{atan2}(r_{21}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = -\pi/2, \;\; \theta_x - \theta_y = \text{atan2}(r_{21}, r_{22}) \tag{12}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{10}$ is tested for the three cases.

```
if (r10 < +1)
{
    if (r10 > -1)
    {
        thetaZ = asin(r10);
        thetaY = atan2(-r20,r00);
        thetaX = atan2(-r12,r11);
    }
    else // r10 = -1
    {
        // Not a unique solution:  thetaX - thetaY = atan2(r21,r22)
        thetaZ = -pi/2;
        thetaY = -atan2(r21,r22);
        thetaX = 0;
    }
}
else
{
    // Not a unique solution:  thetaX + thetaY = atan2(r21,r22)
```

```
    thetaZ = +pi/2;
    thetaY = atan2(r21,r22);
    thetaX = 0;
}
```

## 2.5 Factor as $R_zR_xR_y$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_z(\theta_z)R_x(\theta_x)R_y(\theta_y)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_yc_z - s_xs_ys_z & -c_xs_z & c_zs_y + c_ys_xs_z \\ c_zs_xs_y + c_ys_z & c_xc_z & -c_yc_zs_x + s_ys_z \\ -c_xs_y & s_x & c_xc_y \end{bmatrix}$$

The simplest term to work with is $s_x = r_{21}$, so $\theta_x = \text{asin}(r_{21})$. There are three cases to consider.

CASE 1: If $\theta_x \in (-\pi/2, \pi/2)$, then $c_x \neq 0$ and $c_x(s_y, c_y) = (-r_{20}, r_{22})$, in which case $\theta_y = \text{atan2}(-r_{20}, r_{22})$, and $c_x(s_z, c_z) = (-r_{01}, r_{11})$, in which case $\theta_z = \text{atan2}(-r_{01}, r_{11})$. In summary,

$$\theta_x = \text{asin}(r_{21}), \;\; \theta_z = \text{atan2}(-r_{01}, r_{11}), \;\; \theta_y = \text{atan2}(-r_{20}, r_{22}) \tag{13}$$

CASE 2: If $\theta_x = \pi/2$, then $s_x = 1$ and $c_z = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{10} & r_{12} \end{bmatrix} = \begin{bmatrix} c_ys_z - s_yc_z & c_zs_y + c_ys_z \\ c_zs_y + c_ys_z & -c_yc_z + s_ys_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_y + \theta_z) & \sin(\theta_y + \theta_z) \\ \sin(\theta_y + \theta_z) & -\cos(\theta_y + \theta_z) \end{bmatrix}$$

Therefore, $\theta_y + \theta_z = \text{atan2}(r_{02}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = +\pi/2, \;\; \theta_y + \theta_z = \text{atan2}(r_{02}, r_{00}) \tag{14}$$

CASE 3: If $\theta_x = -\pi/2$, then $s_x = -1$ and $c_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{10} & r_{12} \end{bmatrix} = \begin{bmatrix} c_ys_z + s_yc_z & c_zs_y - c_ys_z \\ -c_zs_y + c_ys_z & c_yc_z + s_ys_z \end{bmatrix} = \begin{bmatrix} \cos(\theta_y - \theta_z) & \sin(\theta_y - \theta_z) \\ -\sin(\theta_y - \theta_z) & \cos(\theta_y - \theta_z) \end{bmatrix}$$

Therefore, $\theta_y - \theta_z = \text{atan2}(r_{02}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = -\pi/2, \;\; \theta_y - \theta_z = \text{atan2}(r_{02}, r_{00}) \tag{15}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{21}$ is tested for the three cases.

```
if (r21 < +1)
{
    if (r21 > -1)
    {
```

```
            thetaX = asin(r21);
            thetaZ = atan2(-r01,r11);
            thetaY = atan2(-r20,r22);
        }
        else // r21 = -1
        {
            // Not a unique solution:  thetaY - thetaZ = atan2(r02,r00)
            thetaX = -pi/2;
            thetaZ = -atan2(r02,r00);
            thetaY = 0;
        }
    }
    else // r21 = +1
    {
        // Not a unique solution:  thetaY + thetaZ = atan2(r02,r00)
        thetaX = +pi/2;
        thetaZ = atan2(r02,r00);
        thetaY = 0;
    }
```

### 2.6 Factor as $R_zR_yR_x$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_yc_z & c_zs_xs_y - c_xs_z & c_xc_zs_y + s_xs_z \\ c_ys_z & c_xc_z + s_xs_ys_z & -c_zs_x + c_xs_ys_z \\ -s_y & c_ys_x & c_xc_y \end{bmatrix}$$

The simplest term to work with is $s_y = -r_{20}$, so $\theta_y = \text{asin}(-r_{20})$. There are three cases to consider.

Case 1: If $\theta_y \in (-\pi/2, \pi/2)$, then $c_y \neq 0$ and $c_y(s_x, c_x) = (r_{21}, r_{22})$, in which case $\theta_x = \text{atan2}(r_{21}, r_{22})$, and $c_y(s_z, c_z) = (r_{10}, r_{00})$, in which case $\theta_z = \text{atan2}(r_{10}, r_{00})$. In summary,

$$\theta_y = \text{asin}(-r_{20}), \;\; \theta_z = \text{atan2}(r_{10}, r_{00}), \;\; \theta_x = \text{atan2}(r_{21}, r_{22}) \tag{16}$$

Case 2: If $\theta_y = \pi/2$, then $s_y = 1$ and $c_y = 0$. In this case,

$$\begin{bmatrix} r_{01} & r_{02} \\ r_{11} & r_{12} \end{bmatrix} = \begin{bmatrix} c_zs_x - c_xs_z & c_xc_z + s_xs_z \\ c_xc_z + s_xs_z & -c_zs_x + c_xs_z \end{bmatrix} = \begin{bmatrix} \sin(\theta_x - \theta_z) & \cos(\theta_x - \theta_z) \\ \cos(\theta_x - \theta_z) & -\sin(\theta_x - \theta_z) \end{bmatrix}$$

Therefore, $\theta_x - \theta_z = \text{atan2}(-r_{12}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = +\pi/2, \;\; \theta_x - \theta_z = \text{atan2}(-r_{12}, r_{11}) \tag{17}$$

CASE 3: If $\theta_y = -\pi/2$, then $s_y = -1$ and $c_y = 0$. In this case,

$$\begin{bmatrix} r_{01} & r_{02} \\ r_{11} & r_{12} \end{bmatrix} = \begin{bmatrix} -c_z s_x - c_x s_z & -c_x c_z + s_x s_z \\ c_x c_z - s_x s_z & -c_z s_x - c_x s_z \end{bmatrix} = \begin{bmatrix} -\sin(\theta_x + \theta_z) & -\cos(\theta_x + \theta_z) \\ \cos(\theta_x + \theta_z) & -\sin(\theta_x + \theta_z) \end{bmatrix}$$

Therefore, $\theta_x + \theta_z = \text{atan2}(-r_{12}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = -\pi/2, \;\; \theta_x + \theta_z = \text{atan2}(-r_{12}, r_{11}) \tag{18}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{20}$ is tested for the three cases.

```
if (r20 < +1)
{
    if (r20 > -1)
    {
        thetaY = asin(-r20);
        thetaZ = atan2(r10,r00);
        thetaX = atan2(r21,r22);
    }
    else // r20 = -1
    {
        // Not a unique solution:  thetaX - thetaZ = atan2(-r12,r11)
        thetaY = +pi/2;
        thetaZ = -atan2(-r12,r11);
        thetaX = 0;
    }
}
else // r20 = +1
{
    // Not a unique solution:  thetaX + thetaZ = atan2(-r12,r11)
    thetaY = -pi/2;
    thetaZ = atan2(-r12,r11);
    thetaX = 0;
}
```

## 2.7 Factor as $R_{x_0} R_y R_{x_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_x(\theta_{x_0}) R_y(\theta_y) R_x(\theta_{x_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y & s_y s_{x_1} & s_y c_{x_1} \\ s_y s_{x_0} & c_{x_0} c_{x_1} - c_y s_{x_0} s_{x_1} & -c_y c_{x_1} s_{x_0} - c_{x_0} s_{x_1} \\ -s_y c_{x_0} & c_{x_1} s_{x_0} + c_y c_{x_0} s_{x_1} & c_y c_{x_0} c_{x_1} - s_{x_0} s_{x_1} \end{bmatrix}$$

The simplest term to work with is $c_y = r_{00}$, so $\theta_y = \text{acos}(r_{00})$. There are three cases to consider.

CASE 1: If $\theta_y \in (0,\pi)$, then $s_y \neq 0$ and $s_y(s_{x_0}, c_{x_0}) = (r_{10}, -r_{20})$, in which case $\theta_{x_0} = \text{atan2}(r_{10}, -r_{20})$, and $s_y(s_{x_1}, c_{x_1}) = (r_{01}, r_{02})$, in which case $\theta_{x_1} = \text{atan2}(r_{01}, r_{02})$. In summary,

$$\theta_y = \text{acos}(r_{00}), \;\; \theta_{x_0} = \text{atan2}(r_{10}, -r_{20}), \;\; \theta_{x_1} = \text{atan2}(r_{01}, r_{02}) \tag{19}$$

CASE 2: If $\theta_y = 0$, then $c_y = 1$ and $s_y = 0$. In this case,

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{x_0}c_{x_1} - s_{x_0}s_{x_1} & -c_{x_1}s_{x_0} - c_{x_0}s_{x_1} \\ c_{x_1}s_{x_0} + c_{x_0}s_{x_1} & c_{x_0}c_{x_1} - s_{x_0}s_{x_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{x_1} + \theta_{x_0}) & -\sin(\theta_{x_1} + \theta_{x_0}) \\ \sin(\theta_{x_1} + \theta_{x_0}) & \cos(\theta_{x_1} + \theta_{x_0}) \end{bmatrix}$$

Therefore, $\theta_{x_1} + \theta_{x_0} = \text{atan2}(-r_{12}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = 0, \;\; \theta_{x_1} + \theta_{x_0} = \text{atan2}(-r_{12}, r_{11}) \tag{20}$$

CASE 3: If $\theta_y = \pi$, then $c_y = -1$ and $s_y = 0$. In this case,

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{x_0}c_{x_1} + s_{x_0}s_{x_1} & c_{x_1}s_{x_0} - c_{x_0}s_{x_1} \\ c_{x_1}s_{x_0} - c_{x_0}s_{x_1} & -c_{x_0}c_{x_1} - s_{x_0}s_{x_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{x_1} - \theta_{x_0}) & -\sin(\theta_{x_1} - \theta_{x_0}) \\ -\sin(\theta_{x_1} - \theta_{x_0}) & -\cos(\theta_{x_1} - \theta_{x_0}) \end{bmatrix}$$

Therefore, $\theta_{x_1} - \theta_{x_0} = \text{atan2}(-r_{12}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = \pi, \;\; \theta_{x_1} - \theta_{x_0} = \text{atan2}(-r_{12}, r_{11}) \tag{21}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{00}$ is tested for the three cases.

```
if (r00 < +1)
{
    if (r00 > -1)
    {
        thetaY = acos(r00);
        thetaX0 = atan2(r10,-r20);
        thetaX1 = atan2(r01,r02);
    }
    else // r00 = -1
    {
        // Not a unique solution:  thetaX1 - thetaX0 = atan2(-r12,r11)
        thetaY = pi;
        thetaX0 = -atan2(-r12,r11);
        thetaX1 = 0;
    }
}
else // r00 = +1
{
    // Not a unique solution:  thetaX1 + thetaX0 = atan2(-r12,r11)
    thetaY = 0;
    thetaX0 = atan2(-r12,r11);
    thetaX1 = 0;
}
```

## 2.8 Factor as $R_{x_0}R_zR_{x_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_x(\theta_{x_0})R_y(\theta_y)R_z(\theta_{x_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_z & -s_z c_{x_1} & s_z s_{x_1} \\ s_z c_{x_0} & c_z c_{x_0} c_{x_1} - s_{x_0} s_{x_1} & -c_{x_1} s_{x_0} - c_z c_{x_0} s_{x_1} \\ s_z s_{x_0} & c_z c_{x_1} s_{x_0} + c_{x_0} s_{x_1} & c_{x_0} c_{x_1} - c_z s_{x_0} s_{x_1} \end{bmatrix}$$

The simplest term to work with is $c_z = r_{00}$, so $\theta_z = \text{acos}(r_{00})$. There are three cases to consider.

CASE 1: If $\theta_z \in (0, \pi)$, then $s_z \neq 0$ and $s_z(s_{x_0}, c_{x_0}) = (r_{20}, r_{10})$, in which case $\theta_{x_0} = \text{atan2}(r_{20}, r_{10})$, and $s_z(s_{x_1}, c_{x_1}) = (r_{02}, -r_{01})$, in which case $\theta_{x_1} = \text{atan2}(r_{02}, -r_{01})$. In summary,

$$\theta_z = \text{acos}(r_{00}), \;\; \theta_{x_0} = \text{atan2}(r_{20}, r_{10}), \;\; \theta_{x_1} = \text{atan2}(r_{02}, -r_{01}) \tag{22}$$

CASE 2: If $\theta_z = 0$, then $c_z = 1$ and $s_z = 0$. In this case,

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{x_0} c_{x_1} - s_{x_0} s_{x_1} & -c_{x_1} s_{x_0} - c_{x_0} s_{x_1} \\ c_{x_1} s_{x_0} + c_{x_0} s_{x_1} & c_{x_0} c_{x_1} - s_{x_0} s_{x_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{x_1} + \theta_{x_0}) & -\sin(\theta_{x_1} + \theta_{x_0}) \\ \sin(\theta_{x_1} + \theta_{x_0}) & \cos(\theta_{x_1} + \theta_{x_0}) \end{bmatrix}$$

Therefore, $\theta_{x_1} + \theta_{x_0} = \text{atan2}(r_{21}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = 0, \;\; \theta_{x_1} + \theta_{x_0} = \text{atan2}(r_{21}, r_{22}) \tag{23}$$

CASE 3: If $\theta_z = \pi$, then $c_z = -1$ and $s_z = 0$. In this case,

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} -c_{x_0} c_{x_1} - s_{x_0} s_{x_1} & -c_{x_1} s_{x_0} + c_{x_0} s_{x_1} \\ -c_{x_1} s_{x_0} + c_{x_0} s_{x_1} & c_{x_0} c_{x_1} + s_{x_0} s_{x_1} \end{bmatrix} = \begin{bmatrix} -\cos(\theta_{x_1} - \theta_{x_0}) & \sin(\theta_{x_1} - \theta_{x_0}) \\ \sin(\theta_{x_1} - \theta_{x_0}) & \cos(\theta_{x_1} - \theta_{x_0}) \end{bmatrix}$$

Therefore, $\theta_{x_1} - \theta_{x_0} = \text{atan2}(r_{21}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = \pi, \;\; \theta_{x_1} - \theta_{x_0} = \text{atan2}(r_{21}, r_{22}) \tag{24}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{00}$ is tested for the three cases.

```
if (r00 < +1)
{
    if (r00 > -1)
    {
        thetaZ = acos(r00);
        thetaX0 = atan2(r20,r10);
        thetaX1 = atan2(r02,-r01);
    }
    else // r00 = -1
    {
```

```
        // Not a unique solution:  thetaX1 - thetaX0 = atan2(r21,r22)
        thetaZ = pi;
        thetaX0 = -atan2(r21,r22);
        thetaX1 = 0;
    }
}
else // r00 = +1
{
    // Not a unique solution:  thetaX1 + thetaX0 = atan2(r21,r22)
    thetaZ = 0;
    thetaX0 = atan2(r21,r22);
    thetaX1 = 0;
}
```

## 2.9 Factor as $R_{y_0}R_xR_{y_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_y(\theta_{y_0})R_x(\theta_x)R_y(\theta_{y_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{y_0}c_{y_1} - c_x s_{y_0} s_{y_1} & s_x s_{y_0} & c_x c_{y_1} s_{y_0} + c_{y_0} s_{y_1} \\ s_x s_{y_1} & c_x & -s_x c_{y_1} \\ -c_{y_1} s_{y_0} - c_x c_{y_0} s_{y_1} & s_x c_{y_0} & c_x c_{y_0} c_{y_1} - s_{y_0} s_{y_1} \end{bmatrix}$$

The simplest term to work with is $c_x = r_{11}$, so $\theta_x = \text{acos}(r_{11})$. There are three cases to consider.

Case 1: If $\theta_x \in (0, \pi)$, then $s_x \neq 0$ and $s_x(s_{y_0}, c_{y_0}) = (r_{01}, r_{21})$, in which case $\theta_{y_0} = \text{atan2}(r_{01}, r_{21})$, and $s_x(s_{y_1}, c_{y_1}) = (r_{10}, -r_{12})$, in which case $\theta_{y_1} = \text{atan2}(r_{10}, -r_{12})$. In summary,

$$\theta_x = \text{acos}(r_{11}), \;\; \theta_{y_0} = \text{atan2}(r_{01}, r_{21}), \;\; \theta_{y_1} = \text{atan2}(r_{10}, -r_{12}) \tag{25}$$

Case 2: If $\theta_x = 0$, then $c_x = 1$ and $s_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{y_0}c_{y_1} - s_{y_0} s_{y_1} & c_{y_1} s_{y_0} + c_{y_0} s_{y_1} \\ -c_{y_1} s_{y_0} - c_{y_0} s_{y_1} & c_{y_0} c_{y_1} - s_{y_0} s_{y_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{y_1} + \theta_{y_0}) & \sin(\theta_{y_1} + \theta_{y_0}) \\ -\sin(\theta_{y_1} + \theta_{y_0}) & \cos(\theta_{y_1} + \theta_{y_0}) \end{bmatrix}$$

Therefore, $\theta_{y_1} + \theta_{y_0} = \text{atan2}(r_{02}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = 0, \;\; \theta_{y_1} + \theta_{y_0} = \text{atan2}(r_{02}, r_{00}) \tag{26}$$

Case 3: If $\theta_x = \pi$, then $c_x = -1$ and $s_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{y_0}c_{y_1} + s_{y_0} s_{y_1} & -c_{y_1} s_{y_0} + c_{y_0} s_{y_1} \\ -c_{y_1} s_{y_0} + c_{y_0} s_{y_1} & -c_{y_0} c_{y_1} - s_{y_0} s_{y_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{y_1} - \theta_{y_0}) & \sin(\theta_{y_1} - \theta_{y_0}) \\ \sin(\theta_{y_1} - \theta_{y_0}) & -\cos(\theta_{y_1} - \theta_{y_0}) \end{bmatrix}$$

Therefore, $\theta_{y_1} - \theta_{y_0} = \text{atan2}(r_{02}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = \pi, \;\; \theta_{y_1} - \theta_{y_0} = \text{atan2}(r_{02}, r_{00}) \tag{27}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{11}$ is tested for the three cases.

```
if (r11 < +1)
{
    if (r11 > -1)
    {
        thetaX = acos(r11);
        thetaY0 = atan2(r01,r21);
        thetaY1 = atan2(r10,-r12);
    }
    else // r11 = -1
    {
        // Not a unique solution:  thetaY1 - thetaY0 = atan2(r02,r00)
        thetaX = pi;
        thetaY0 = -atan2(r02,r00);
        thetaY1 = 0;
    }
}
else // r11 = +1
{
    // Not a unique solution:  thetaY1 + thetaY0 = atan2(r02,r00)
    thetaX = 0;
    thetaY0 = atan2(r02,r00);
    thetaY1 = 0;
}
```

## 2.10 Factor as $R_{y_0}R_zR_{y_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_y(\theta_{y_0})R_z(\theta_z)R_y(\theta_{y_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_z c_{y_0} c_{y_1} - s_{y_0} s_{y_1} & -s_z c_{y_0} & c_{y_1} s_{y_0} + c_z c_{y_0} s_{y_1} \\ s_z c_{y_1} & c_z & s_z s_{y_1} \\ -c_z c_{y_1} s_{y_0} - c_{y_0} s_{y_1} & s_z s_{y_0} & c_{y_0} c_{y_1} - c_z s_{y_0} s_{y_1} \end{bmatrix}$$

The simplest term to work with is $c_z = r_{11}$, so $\theta_z = \text{acos}(r_{11})$. There are three cases to consider.

Case 1: If $\theta_z \in (0, \pi)$, then $s_z \neq 0$ and $s_z(s_{y_0}, c_{y_0}) = (r_{21}, -r_{01})$, in which case $\theta_{y_0} = \text{atan2}(r_{21}, -r_{01})$, and $s_z(s_{y_1}, c_{y_1}) = (r_{12}, r_{10})$, in which case $\theta_{y_1} = \text{atan2}(r_{12}, r_{10})$. In summary,

$$\theta_z = \text{acos}(r_{11}), \;\; \theta_{y_0} = \text{atan2}(r_{21}, -r_{01}), \;\; \theta_{y_1} = \text{atan2}(r_{12}, r_{10}) \tag{28}$$

Case 2: If $\theta_z = 0$, then $c_z = 1$ and $s_z = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{y_0} c_{y_1} - s_{y_0} s_{y_1} & c_{y_1} s_{y_0} + c_{y_0} s_{y_1} \\ -c_{y_1} s_{y_0} - c_{y_0} s_{y_1} & c_{y_0} c_{y_1} - s_{y_0} s_{y_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{y_1} + \theta_{y_0}) & \sin(\theta_{y_1} + \theta_{y_0}) \\ -\sin(\theta_{y_1} + \theta_{y_0}) & \cos(\theta_{y_1} + \theta_{y_0}) \end{bmatrix}$$

Therefore, $\theta_{y_1} + \theta_{y_0} = \text{atan2}(-r_{20}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = 0, \;\; \theta_{y_1} + \theta_{y_0} = \text{atan2}(-r_{20}, r_{22}) \tag{29}$$

Case 3: If $\theta_z = \pi$, then $c_z = -1$ and $s_z = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{02} \\ r_{20} & r_{22} \end{bmatrix} = \begin{bmatrix} -c_{y_0}c_{y_1} - s_{y_0}s_{y_1} & c_{y_1}s_{y_0} - c_{y_0}s_{y_1} \\ c_{y_1}s_{y_0} - c_{y_0}s_{y_1} & c_{y_0}c_{y_1} + s_{y_0}s_{y_1} \end{bmatrix} = \begin{bmatrix} -\cos(\theta_{y_1} - \theta_{y_0}) & -\sin(\theta_{y_1} - \theta_{y_0}) \\ -\sin(\theta_{y_1} - \theta_{y_0}) & \cos(\theta_{y_1} - \theta_{y_0}) \end{bmatrix}$$

Therefore, $\theta_{y_1} - \theta_{y_0} = \text{atan2}(-r_{20}, r_{22})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_z = \pi, \;\; \theta_{y_1} - \theta_{y_0} = \text{atan2}(-r_{20}, r_{22}) \tag{30}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{11}$ is tested for the three cases.

```
if (r11 < +1)
{
    if (r11 > -1)
    {
        thetaZ = acos(r11);
        thetaY0 = atan2(r21,-r01);
        thetaY1 = atan2(r12,r10);
    }
    else // r11 = -1
    {
        // Not a unique solution:  thetaY1 - thetaY0 = atan2(-r20,r22)
        thetaZ = pi;
        thetaY0 = -atan2(-r20,r22);
        thetaY1 = 0;
    }
}
else // r11 = +1
{
    // Not a unique solution:  thetaY1 + thetaY0 = atan2(-r20,r22)
    thetaZ = 0;
    thetaY0 = atan2(-r20,r22);
    thetaY1 = 0;
}
```

## 2.11 Factor as $R_{z_0}R_xR_{z_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_z(\theta_{z_0})R_x(\theta_x)R_z(\theta_{z_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_{z_0}c_{z_1} - c_x s_{z_0} s_{z_1} & -c_x c_{z_1} s_{z_0} - c_{z_0} s_{z_1} & s_x s_{z_0} \\ c_{z_1} s_{z_0} + c_x c_{z_0} s_{z_1} & c_x c_{z_0} c_{z_1} - s_{z_0} s_{z_1} & -s_x c_{z_0} \\ s_x s_{z_1} & s_x c_{z_1} & c_x \end{bmatrix}$$

The simplest term to work with is $c_x = r_{22}$, so $\theta_x = \text{acos}(r_{22})$. There are three cases to consider.

Case 1: If $\theta_x \in (0, \pi)$, then $s_x \neq 0$ and $s_x(s_{z_0}, c_{z_0}) = (r_{02}, -r_{12})$, in which case $\theta_{z_0} = \text{atan2}(r_{02}, -r_{12})$, and $s_x(s_{z_1}, c_{z_1}) = (r_{20}, r_{21})$, in which case $\theta_{z_1} = \text{atan2}(r_{20}, r_{21})$. In summary,

$$\theta_x = \text{acos}(r_{22}), \;\; \theta_{z_0} = \text{atan2}(r_{02}, -r_{12}), \;\; \theta_{z_1} = \text{atan2}(r_{20}, r_{21}) \tag{31}$$

Case 2: If $\theta_x = 0$, then $c_x = 1$ and $s_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{bmatrix} = \begin{bmatrix} c_{z_0}c_{z_1} - s_{z_0} s_{z_1} & -c_{z_1} s_{z_0} - c_{z_0} s_{z_1} \\ c_{z_1} s_{z_0} + c_{z_0} s_{z_1} & c_{z_0} c_{z_1} - s_{z_0} s_{z_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{z_1} + \theta_{z_0}) & -\sin(\theta_{z_1} + \theta_{z_0}) \\ \sin(\theta_{z_1} + \theta_{z_0}) & \cos(\theta_{z_1} + \theta_{z_0}) \end{bmatrix}$$

Therefore, $\theta_{z_1} + \theta_{z_0} = \text{atan2}(-r_{01}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = 0, \;\; \theta_{z_1} + \theta_{z_0} = \text{atan2}(-r_{01}, r_{00}) \tag{32}$$

Case 3: If $\theta_x = \pi$, then $c_x = -1$ and $s_x = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{bmatrix} = \begin{bmatrix} c_{z_0}c_{z_1} + s_{z_0} s_{z_1} & c_{z_1} s_{z_0} - c_{z_0} s_{z_1} \\ c_{z_1} s_{z_0} - c_{z_0} s_{z_1} & -c_{z_0} c_{z_1} - s_{z_0} s_{z_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{z_1} - \theta_{z_0}) & -\sin(\theta_{z_1} - \theta_{z_0}) \\ -\sin(\theta_{z_1} - \theta_{z_0}) & -\cos(\theta_{z_1} - \theta_{z_0}) \end{bmatrix}$$

Therefore, $\theta_{z_1} - \theta_{z_0} = \text{atan2}(-r_{01}, r_{00})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_x = \pi, \;\; \theta_{z_1} - \theta_{z_0} = \text{atan2}(-r_{01}, r_{00}) \tag{33}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{22}$ is tested for the three cases.

```
if (r22 < +1)
{
    if (r22 > -1)
    {
        thetaX = acos(r22);
        thetaZ0 = atan2(r02,-r12);
        thetaZ1 = atan2(r20,r21);
    }
    else // r22 = -1
    {
```

```
            // Not a unique solution:  thetaZ1 - thetaZ0 = atan2(-r01,r00)
            thetaX = pi;
            thetaZ0 = -atan2(-r01,r00);
            thetaZ1 = 0;
        }
    }
    else // r22 = +1
    {
        // Not a unique solution:  thetaZ1 + thetaZ0 = atan2(-r01,r00)
        thetaX = 0;
        thetaZ0 = atan2(-r01,r00);
        thetaZ1 = 0;
    }
```

## 2.12 Factor as $R_{z_0}R_yR_{z_1}$

Setting $R = [r_{ij}]$ for $0 \le i \le 2$ and $0 \le j \le 2$, formally multiplying $R_z(\theta_{z_0})R_y(\theta_y)R_z(\theta_{z_1})$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_{z_0} c_{z_1} - s_{z_0} s_{z_1} & -c_{z_1} s_{z_0} - c_y c_{z_0} s_{z_1} & s_y c_{z_0} \\ c_y c_{z_1} s_{z_0} + c_{z_0} s_{z_1} & c_{z_0} c_{z_1} - c_y s_{z_0} s_{z_1} & s_y s_{z_0} \\ -s_y c_{z_1} & s_y s_{z_1} & c_y \end{bmatrix}$$

The simplest term to work with is $c_y = r_{22}$, so $\theta_y = \text{acos}(r_{22})$. There are three cases to consider.

Case 1: If $\theta_y \in (0, \pi)$, then $s_y \neq 0$ and $s_y(s_{z_0}, c_{z_0}) = (r_{12}, r_{02})$, in which case $\theta_{z_0} = \text{atan2}(r_{12}, r_{02})$, and $s_y(s_{z_1}, c_{z_1}) = (r_{21}, -r_{20})$, in which case $\theta_{z_1} = \text{atan2}(r_{20}, r_{21})$. In summary,

$$\theta_y = \text{acos}(r_{22}), \;\; \theta_{z_0} = \text{atan2}(r_{12}, r_{02}), \;\; \theta_{z_1} = \text{atan2}(r_{21}, -r_{-20}) \tag{34}$$

Case 2: If $\theta_y = 0$, then $c_y = 1$ and $s_y = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{bmatrix} = \begin{bmatrix} c_{z_0} c_{z_1} - s_{z_0} s_{z_1} & -c_{z_1} s_{z_0} - c_{z_0} s_{z_1} \\ c_{z_1} s_{z_0} + c_{z_0} s_{z_1} & c_{z_0} c_{z_1} - s_{z_0} s_{z_1} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{z_1} + \theta_{z_0}) & -\sin(\theta_{z_1} + \theta_{z_0}) \\ \sin(\theta_{z_1} + \theta_{z_0}) & \cos(\theta_{z_1} + \theta_{z_0}) \end{bmatrix}$$

Therefore, $\theta_{z_1} + \theta_{z_0} = \text{atan2}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = 0, \;\; \theta_{z_1} + \theta_{z_0} = \text{atan2}(r_{10}, r_{11}) \tag{35}$$

Case 3: If $\theta_y = \pi$, then $c_y = -1$ and $s_y = 0$. In this case,

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{bmatrix} = \begin{bmatrix} -c_{z_0} c_{z_1} - s_{z_0} s_{z_1} & -c_{z_1} s_{z_0} + c_{z_0} s_{z_1} \\ -c_{z_1} s_{z_0} + c_{z_0} s_{z_1} & c_{z_0} c_{z_1} + s_{z_0} s_{z_1} \end{bmatrix} = \begin{bmatrix} -\cos(\theta_{z_1} + \theta_{z_0}) & -\sin(\theta_{z_1} + \theta_{z_0}) \\ \sin(\theta_{z_1} + \theta_{z_0}) & \cos(\theta_{z_1} + \theta_{z_0}) \end{bmatrix}$$

Therefore, $\theta_{z_1} - \theta_{z_0} = \text{atan2}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. In summary,

$$\theta_y = \pi, \;\; \theta_{z_1} - \theta_{z_0} = \text{atan2}(r_{10}, r_{11}) \tag{36}$$

Pseudocode for the factorization is listed below. To avoid the arcsin call until needed, the matrix entry $r_{22}$ is tested for the three cases.

```
if (r22 < +1)
{
    if (r22 > -1)
    {
        thetaY = acos(r22);
        thetaZ0 = atan2(r12,r02);
        thetaZ1 = atan2(r21,-r20);
    }
    else // r22 = -1
    {
        // Not a unique solution:  thetaZ1 - thetaZ0 = atan2(r10,r11)
        thetaY = pi;
        thetaZ0 = -atan2(r10,r11);
        thetaZ1 = 0;
    }
}
else // r22 = +1
{
    // Not a unique solution:  thetaZ1 + thetaZ0 = atan2(r10,r11)
    thetaY = 0;
    thetaZ0 = atan2(r10,r11);
    thetaZ1 = 0;
}
```

# 3 Factor as a Product of Two Rotation Matrices

Given a rotation $R$ that is a product of two coordinate axis rotations, the problem is to factor it into three coordinate axis rotations using the ordering $xyz$. Derivations for the other orderings are similar. In the subsections the matrices are $P_x = R_x(\phi_x)$, $P_y = R_y(\phi_y)$, and $P_z = R_z(\phi_z)$. Define $s_a = \sin(\phi_x)$, $s_b = \sin(\phi_y)$, $s_c = \sin(\phi_z)$, $c_a = \cos(\phi_x)$, $c_b = \cos(\phi_y)$, and $c_c = \cos(\phi_z)$.

## 3.1 Factor $P_xP_y$

This is a trivial case. The factorization is $R = R_x(\phi_x)R_y(\phi_y) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = \phi_x$, $\theta_y = \phi_y$, and $\theta_z = 0$.

### 3.2 Factor $P_yP_x$

The factorization is $R = R_y(\phi_y)R_x(\phi_x) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Formal multiplication of the various terms leads to the equation

$$\begin{bmatrix} c_b & s_a s_b & c_a s_b \\ 0 & c_a & -s_a \\ -s_b & c_b s_a & c_a c_b \end{bmatrix} = \begin{bmatrix} c_y c_z & -c_y s_z & s_y \\ c_z s_x s_y + c_x s_z & c_x c_z - s_x s_y s_z & -c_y s_x \\ -c_x c_z s_y + s_x s_z & c_z s_x + c_x s_y s_z & c_x c_y \end{bmatrix}.$$

It is easy to see that $s_y = c_a s_b$ in which case $\theta_y = \text{asin}(\cos\theta_y \sin\theta_x)$. Adding the 10 and 21 terms yields

$$0 + c_b s_a = (c_z s_x s_y + c_x s_z) + (c_z s_x + c_x s_y s_z) = (1 + s_y)(c_z s_x + c_x s_z)$$

which leads to $\sin(\theta_x + \theta_z) = c_b s_a/(1 + c_a s_b)$. In the even that $c_a s_b = -1$, this leads to a special case in the coding that is easy to solve. Subtracting the 10 term from the 21 term yields

$$c_b s_a - 0 = (c_z s_x s_y + c_x s_z) - (c_z s_x + c_x s_y s_z) = (1 - s_y)(c_z s_x - c_x s_z)$$

which leads to $\sin(\theta_x - \theta_z) = c_b s_a/(1 - c_a s_b)$. In the event that $c_a s_b = 1$, this also leads to a special case in the coding that is easy to solve. The sine functions can be inverted and the two resulting equations for $\theta_x$ and $\theta_z$ can be solved. For the case $|c_a s_b| < 1$,

$$\begin{aligned} \theta_x &= \tfrac{1}{2}\left[\text{asin}\left(\tfrac{c_b s_a}{1+c_b s_a}\right) + \text{asin}\left(\tfrac{c_b s_a}{1-c_b s_a}\right)\right] \\ \theta_y &= \text{asin}(c_a s_b) \\ \theta_z &= \tfrac{1}{2}\left[\text{asin}\left(\tfrac{c_b s_a}{1+c_b s_a}\right) - \text{asin}\left(\tfrac{c_b s_a}{1-c_b s_a}\right)\right] \end{aligned}$$

### 3.3 Factor $P_xP_z$

This is a trivial case. The factorization is $R = R_x(\phi_x)R_z(\phi_z) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = \phi_x$, $\theta_y = 0$, and $\theta_z = \phi_z$.

### 3.4 Factor $P_zP_x$

A construction similar to the case $P_yP_x$ leads to

$$\begin{aligned} \theta_x &= \tfrac{1}{2}\left[\text{asin}\left(\tfrac{c_a c_c}{1+s_a s_c}\right) + \text{asin}\left(\tfrac{c_a c_c}{1-s_a s_c}\right)\right] \\ \theta_y &= \text{asin}(s_a s_c) \\ \theta_z &= \tfrac{1}{2}\left[\text{asin}\left(\tfrac{c_a c_c}{1+s_a s_c}\right) - \text{asin}\left(\tfrac{c_a c_c}{1-s_a s_c}\right)\right] \end{aligned}$$

### 3.5 Factor $P_yP_z$

This is a trivial case. The factorization is $R = R_y(\phi_y)R_z(\phi_z) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = 0$, $\theta_y = \phi_y$, and $\theta_z = \phi_z$.

### 3.6 Factor $P_z P_y$

A construction similar to the case $P_y P_x$ leads to

$$\begin{aligned}
\theta_x &= \tfrac{1}{2}\left[\operatorname{asin}\left(\tfrac{c_b s_c}{1+s_b c_c}\right) - \operatorname{asin}\left(\tfrac{c_b s_c}{1-s_b c_c}\right)\right] \\
\theta_y &= \operatorname{asin}(s_b c_c) \\
\theta_z &= \tfrac{1}{2}\left[\operatorname{asin}\left(\tfrac{c_b s_c}{1+s_b c_c}\right) + \operatorname{asin}\left(\tfrac{c_b s_c}{1-s_b c_c}\right)\right]
\end{aligned}$$