Geodesics on a Parameterized Surface

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1 Discussion

Consider a parameterized surface $\mathbf{x}(u,v)$. Let $\mathbf{x}_u$ and $\mathbf{x}_v$ denote the first-order partial derivatives. Let $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$ denote the second-order partial derivatives. From differential geometry, the metric for the surface is

$$G = [g_{ij}] = \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_u \cdot \mathbf{x}_v & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix}$$

The inverse of the matrix is

$$G^{-1} = [g^{ij}] = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} \begin{bmatrix} \mathbf{x}_v \cdot \mathbf{x}_v & -\mathbf{x}_u \cdot \mathbf{x}_v \\ -\mathbf{x}_u \cdot \mathbf{x}_v & -\mathbf{x}_u \cdot \mathbf{x}_u \end{bmatrix}$$

The Christoffel symbols of the first kind are Γ_{ijk} and are symmetric in i and j. They are

$$\begin{aligned} \Gamma_{111} &= \mathbf{x}_{uu} \cdot \mathbf{x}_u \\ \Gamma_{121} &= \mathbf{x}_{uv} \cdot \mathbf{x}_u \\ \Gamma_{221} &= \mathbf{x}_{vv} \cdot \mathbf{x}_u \\ \Gamma_{112} &= \mathbf{x}_{uu} \cdot \mathbf{x}_v \\ \Gamma_{122} &= \mathbf{x}_{uv} \cdot \mathbf{x}_v \\ \Gamma_{222} &= \mathbf{x}_{vv} \cdot \mathbf{x}_v \end{aligned}$$

The Christoffel symbols of the second kind are Γ^i_{jk} and are symmetric in i and j. They are

$$\begin{aligned} \Gamma^1_{11} &= g^{11}\Gamma_{111} + g^{12}\Gamma_{112} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_v\right] \\ \Gamma^1_{12} &= g^{11}\Gamma_{121} + g^{12}\Gamma_{122} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_v\right] \\ \Gamma^1_{22} &= g^{11}\Gamma_{221} + g^{12}\Gamma_{222} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_v\right] \\ \Gamma^2_{11} &= g^{21}\Gamma_{111} + g^{22}\Gamma_{112} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{uu} \cdot \mathbf{x}_v\right] \\ \Gamma^2_{12} &= g^{21}\Gamma_{121} + g^{22}\Gamma_{122} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{uv} \cdot \mathbf{x}_v\right] \\ \Gamma^2_{22} &= g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = \tfrac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2}\left[-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{vv} \cdot \mathbf{x}_v\right] \end{aligned}$$

A geodesic curve must satisfy the following system of differential equations in parameter space where s is the arc length parameter,

$$\begin{aligned} \tfrac{d^2u}{ds} + \Gamma^1_{11}\left(\tfrac{du}{ds}\right)^2 + 2\Gamma^1_{12}\tfrac{du}{ds}\tfrac{dv}{ds} + \Gamma^1_{22}\left(\tfrac{dv}{ds}\right)^2 = 0 \\ \tfrac{d^2v}{ds} + \Gamma^2_{11}\left(\tfrac{du}{ds}\right)^2 + 2\Gamma^2_{12}\tfrac{du}{ds}\tfrac{dv}{ds} + \Gamma^2_{22}\left(\tfrac{dv}{ds}\right)^2 = 0 \end{aligned}$$

The curve itself is $\mathbf{y}(s) = \mathbf{x}(u(s), v(s))$. Initial conditions specified on the surface are $\mathbf{y}_0 = \mathbf{y}(0)$ and $\mathbf{y}_1 = \mathbf{y}'(0)$ with $|\mathbf{y}_1| = 1$. The Jacobian matrix is the 3×2 matrix whose columns are $\mathbf{x}_u$ and $\mathbf{x}_v$. Note that $J^T J = G$. The derivative in parameter space is

$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = G^{-1} J^T \mathbf{y}'(s).$$

Given $\mathbf{y}'(0)$, the corresponding initial derivative in parameter space can be computed.

The system of equations is of the form

$$u'' = F(u, v, u', v')$$
$$v'' = G(u, v, u', v')$$

where the $'$ indicate derivatives. Setting $p = u'$ and $q = v'$, this becomes a system of four equations

$$\begin{aligned} u' &= p \\ v' &= q \\ p' &= F(u, v, p, q) \\ q' &= G(u, v, p, q) \end{aligned}$$

The initial value problem can be solved using standard differential equation solvers such as Runge-Kutta 4th order.