The Laplace Expansion Theorem: Computing the Determinants and Inverses of Matrices

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Contents

A standard method for symbolically computing the determinant of an $n \times n$ matrix involves *cofactors* and *expanding by a row or by a column*. This document describes the standard formulas for computing the determinants of 2×2 and 3×3 matrices, mentions the general form of *Laplace Expansion Theorem* for which the standard determinant formulas are special cases, and shows how to compute the determinant of a 4×4 matrix using (1) expansion by a row or column and (2) expansion by 2×2 submatrices. Method (2) involves fewer arithmetic operations than does method (1).

1 Determinants and Inverses of 2×2 Matrices

The prototypical example is for a 2×2 matrix, $A = [a_{rc}]$, where the row index satisfies $0 \le r \le 1$ and the column index satisfies $0 \le c \le 1$. The matrix is

$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$

Expanding by the first row,

$$\det(A) = +a_{00} \cdot \det[a_{11}] - a_{01} \cdot \det[a_{10}] = a_{00}a_{11} - a_{01}a_{10} \tag{1}$$

where the determinant of a 1×1 matrix is just the single entry of that matrix. The terms in the determinant formula for a 2×2 matrix involve the matrix entries in the first row, an alternating sign for these entries, and determinants of 1×1 *submatrices*. For example, the first term in the formula uses row entry a_{00}, sign $+1$, and submatrix $[a_{11}]$. The row entry a_{00} has row index 0 and column index 0. The submatrix $[a_{11}]$ is obtained from A by deleting row 0 and column 0. The second term in the formula uses row entry a_{01}, sign -1, and submatrix $[a_{10}]$. The row entry a_{01} has row index 0 and column index 1. The submatrix $[a_{10}]$ is obtained from A by deleting row 0 and column 1.

Similarly, you may expand by the second row:

$$\det(A) = -a_{10} \cdot \det[a_{01}] + a_{11} \det[a_{00}] = -a_{10}a_{01} + a_{11}a_{00} \tag{2}$$

The first term in the formula uses row entry a_{10}, sign -1, and submatrix $[a_{01}]$. The row entry a_{10} has row index 1 and column index 0. The submatrix $[a_{01}]$ is obtained from A by deleting row 1 and column 0. The second term in the formula uses row entry a_{11}, sign $+1$, and submatrix $[a_{00}]$. The row entry a_{11} has row index 1 and column index 1. The submatrix $[a_{00}]$ is obtained from A by deleting row 1 and column 1.

Expansions by column are also possible. Expanding by the first column leads to

$$\det(A) = +a_{00} \cdot \det[a_{11}] - a_{10} \cdot \det[a_{01}] = a_{00}a_{11} - a_{10}a_{01} \tag{3}$$

and expanding by the second column leads to

$$\det(A) = -a_{01} \cdot \det[a_{10}] + a_{11} \cdot \det[a_{00}] = -a_{01}a_{10} + a_{11}a_{00} \tag{4}$$

The four determinant formulas, Equations (1) through (4), are examples of the Laplace Expansion Theorem. The sign associated with an entry a_{rc} is $(-1)^{r+c}$. For example, in expansion by the first row, the sign associated with a_{00} is $(-1)^{0+0} = 1$ and the sign associated with a_{01} is $(-1)^{0+1} = -1$. A determinant of a submatrix $[a_{rc}]$ is called a *minor*. The combination of the sign and minor in a term of the determinant

formula is called a *cofactor* for the matrix entry that occurred in the term. For example, in the second term of Equation (1), the sign is -1, the minor is $\det[a_{10}]$, and the cofactor is $-a_{10}$. This cofactor is associated with the matrix entry a_{01}. The cofactors may be stored in a matrix called the *adjugate* of A,

$$\operatorname{adj}(A) = \begin{bmatrix} +a_{11} & -a_{10} \\ -a_{01} & +a_{00} \end{bmatrix} \tag{5}$$

This matrix has the property

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I \tag{6}$$

where I is the 2×2 identity matrix. When $\det(A)$ is not zero, the matrix A has an inverse given by

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) \tag{7}$$

2 Determinants and Inverses of 3×3 Matrices

Consider a 3×3 matrix $A = [a_{rc}]$, where the row index satisfies $0 \le r \le 2$ and the column index satisfies $0 \le c \le 2$. The matrix is

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

Expanding by the first row,

$$\begin{aligned} \det(A) &= +a_{00} \cdot \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - a_{01} \cdot \det \begin{bmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{bmatrix} + a_{02} \cdot \det \begin{bmatrix} a_{10} & a_{11} \\ a_{20} & a_{21} \end{bmatrix} \\ &= +a_{00}(a_{11}a_{22} - a_{12}a_{21}) - a_{01}(a_{10}a_{22} - a_{12}a_{20}) + a_{02}(a_{10}a_{21} - a_{11}a_{20}) \\ &= +a_{00}a_{11}a_{22} + a_{01}a_{12}a_{20} + a_{02}a_{10}a_{21} - a_{00}a_{12}a_{21} - a_{01}a_{10}a_{22} - a_{02}a_{11}a_{20} \end{aligned} \tag{8}$$

Each term in the first line of Equation (8) involves a sign, an entry from row 0 of A, and a determinant of a submatrix of A. If a_{0c} is an entry in row 0, then the sign is $(-1)^{0+c}$ and the submatrix is obtained by removing row 0 and column c from A.

Five other expansions produce the same determinant formula: by row 1, by row 2, by column 0, by column 1, or by column 2. In all six formulas, each term involves a matrix entry a_{rc}, an associated sign $(-1)^{r+c}$, and a submatrix M_{rc} that is obtained from A by removing row r and column c. The cofactor associated with the term is

$$\gamma_{rc} = (-1)^{r+c} \det M_{rc}$$

The matrix of cofactors is $\Gamma = [\gamma_{rc}]$ for rows $0 \le r \le 2$ and for columns $0 \le c \le 2$. The transpose of the matrix of cofactors is called the adjugate matrix, denoted $\operatorname{adj}(A)$, and as in the 2×2 case, satisfies Equation (6). When the determinant is not zero, the inverse of A is defined by Equation (7). In the case of the 3×3

matrix, the adjugate is

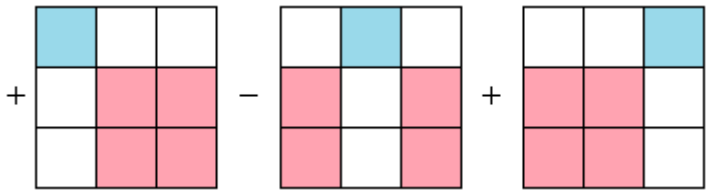
$$\text{adj}(A) = \begin{bmatrix} +(a_{11}a_{22} - a_{12}a_{21}) & -(a_{01}a_{22} - a_{02}a_{21}) & +(a_{01}a_{12} - a_{02}a_{11}) \\ -(a_{10}a_{22} - a_{12}a_{20}) & +(a_{00}a_{22} - a_{02}a_{20}) & -(a_{00}a_{12} - a_{02}a_{10}) \\ +(a_{10}a_{21} - a_{11}a_{20}) & -(a_{00}a_{21} - a_{01}a_{20}) & +(a_{00}a_{11} - a_{01}a_{10}) \end{bmatrix} \tag{9}$$

The first line of Equation (8) may be written also as

$$\det(A) = +\det[a_{00}] \cdot \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \det[a_{01}] \cdot \det \begin{bmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{bmatrix} + \det[a_{02}] \cdot \det \begin{bmatrix} a_{10} & a_{11} \\ a_{20} & a_{21} \end{bmatrix} \tag{10}$$

which is a sum of products of determinant of submatrices, with alternating signs for the terms. A visual way to look at this is shown in Figure (2.1).

Figure 2.1 A visualization of the determinant of a 3×3 matrix.



Each 3×3 grid represents the matrix entries. The blue-colored cells represent the 1×1 submatrices in the determinant formula and the red-colored cells represent the 2×2 submatrices in the determinant formula.

In the left 3×3 grid of the figure, the blue-colored cell represents the submatrix $[a_{00}]$ from the first term in the determinant formula. The red-colored cells are the *complementary submatrix* of $[a_{00}]$, namely, the 2×2 submatrix that is part of the first term of the formula: the first row has a_{11} and a_{12} and the second row has a_{21} and a_{22}. The submatrix is obtained from A by removing row 0 and column 0.

In the middle 3×3 grid of the figure, the blue-colored cell represents the submatrix $[a_{01}]$ from the second term in the determinant formula. The red-colored cells are the complementary submatrix of $[a_{01}]$, namely, the 2×2 submatrix that is part of the second term of the formula: the first row has a_{10} and a_{12} and the second row has a_{20} and a_{22}. The submatrix is obtained from A by removing row 0 and column 1.

In the right 3×3 grid of the figure, the blue-colored cell represents the submatrix $[a_{02}]$ from the third term in the determinant formula. The red-colored cells are the complementary submatrix of $[a_{02}]$, namely, the 2×2 matrix that is part of the third term of the formula: the first row has a_{10} and a_{11} and the second row has a_{20} and a_{21}. The submatrix is obtained from A by removing row 0 and column 2.

3 The Laplace Expansion Theorem

This theorem is a very general formula for computing the determinant of an $n \times n$ matrix A. First, some definitions. Let $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ be a list of k row indices for A, where $1 \leq k < n$ and $0 \leq r_1 < r_2 < \cdots < r_k < n$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a list of k column indices for A, where $1 \leq k < n$ and $0 \leq c_1 < c_2 < \cdots < c_k < n$. The submatrix obtained by *keeping* the entries in the intersection of any row and column that are in the lists is denoted

$$S(A; \mathbf{r}, \mathbf{c}) \tag{11}$$

The submatrix obtained by *removing* the entries in the rows and columns that are in the list is denoted

$$S'(A; \mathbf{r}, \mathbf{c}) \tag{12}$$

and is the *complementary submatrix* for $S(A; \mathbf{r}, \mathbf{c})$.

For example, let A be a 3×3 matrix. Let $\mathbf{r} = (0)$ and $\mathbf{c} = (1)$. Then

$$S(A; \mathbf{r}, \mathbf{c}) = [a_{01}], \quad S'(A; \mathbf{r}, \mathbf{c}) = \begin{bmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{bmatrix}$$

In the middle 3×3 grid of Figure 2.1, $S(A; (0), (1))$ is formed from the blue-colored cell and $S'(A; (0), (1))$ is formed from the red-colored cells.

Laplace Expansion Theorem. Let A be an $n \times n$ matrix. Let $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ be a list of k row indices, where $1 \leq k < n$ and $0 \leq r_1 < r_2 < \cdots r_k < n$. The determinant of A is

$$\det(A) = (-1)^{|\mathbf{r}|} \sum_{\mathbf{c}} (-1)^{|\mathbf{c}|} \det S(A; \mathbf{r}, \mathbf{c}) \det S'(A; \mathbf{r}, \mathbf{c}) \tag{13}$$

where $|\mathbf{r}| = r_1 + r_2 + \cdots + r_k$, $|\mathbf{c}| = c_1 + c_2 + \cdots + c_k$, and the summation is over all k-tuples $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ for which $1 \leq c_1 < c_2 < \cdots < c_k < n$. $\bowtie$

For example, consider a 3×3 matrix with $\mathbf{r} = (0)$ (that is, $k = 1$). Then $|\mathbf{r}| = 0$, $\mathbf{c} = (c_0)$, and the determinant is

$$\begin{aligned}
\det(A) &= \textstyle\sum_{c_0=0}^{2} (-1)^{c_0} \det S(A; (0), (c_0)) \det S'(A; (0), (c_0)) \\
&= (-1)^0 \det S(A; (0), (0)) \det S'(A; (0), (0)) + (-1)^1 \det S(A; (0), (1)) \det S'(A; (0), (1)) \\
&\quad + (-1)^2 \det S(A; (0), (2)) \det S'(A; (0), (2)) \\
&= +\det[a_{00}] \cdot \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \det[a_{01}] \cdot \det \begin{bmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{bmatrix} + \det[a_{02}] \cdot \det \begin{bmatrix} a_{10} & a_{11} \\ a_{20} & a_{21} \end{bmatrix}
\end{aligned}$$

which is Equation (10).

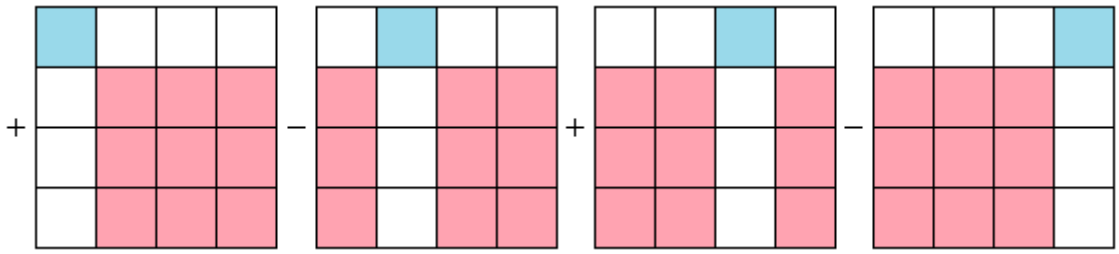
4 Determinants and Inverses of 4×4 Matrices

The Laplace Expansion Theorem may be applied to 4×4 matrices in a couple of ways. The first way uses an expansion by a row or by a column, which is what most people are used to doing. The matrix is

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Using the visualization as motivated by Figure 2.1, an expansion by row 0 is visualized in Figure 4.1:

Figure 4.1 A visualization of the expansion by row 0 of a 4×4 matrix in order to compute the determinant.



The algebraic equivalent is

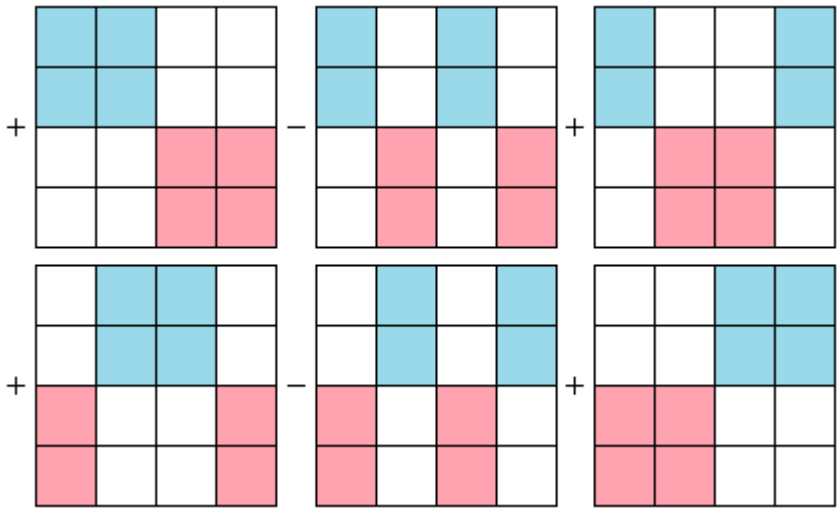
$$\begin{aligned} \det(A) \quad = \quad & +\det[a_{00}] \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \det[a_{01}] \cdot \det \begin{bmatrix} a_{10} & a_{12} & a_{13} \\ a_{20} & a_{22} & a_{23} \\ a_{30} & a_{32} & a_{33} \end{bmatrix} \\ & + \det[a_{02}] \cdot \det \begin{bmatrix} a_{10} & a_{11} & a_{13} \\ a_{20} & a_{21} & a_{23} \\ a_{30} & a_{31} & a_{33} \end{bmatrix} - \det[a_{03}] \cdot \det \begin{bmatrix} a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{bmatrix} \end{aligned} \tag{14}$$

It is possible, however, to use the Laplace Expansion Theorem in a different manner. Choose $\mathbf{r} = (0, 1)$, an expansion by rows 0 and 1, so to speak; then $|\mathbf{r}| = 0 + 1 = 1$, $\mathbf{c} = (c_0, c_1)$, and

$$\begin{aligned}
\det(A) &= -\textstyle\sum_{(c_0,c_1)} (-1)^{c_0+c_1} \det S(A; (0,1), (c_0, c_1)) \det S'(A; (0,1), (c_0, c_1)) \\
&= + \det S(A; (0,1), (0,1)) \det S'(A; (0,1), (0,1)) \\
&\quad - \det S(A; (0,1), (0,2)) \det S'(A; (0,1), (0,2)) \\
&\quad + \det S(A; (0,1), (0,3)) \det S'(A; (0,1), (0,3)) \\
&\quad + \det S(A; (0,1), (1,2)) \det S'(A; (0,1), (1,2)) \\
&\quad - \det S(A; (0,1), (1,3)) \det S'(A; (0,1), (1,3)) \\
&\quad + \det S(A; (0,1), (2,3)) \det S'(A; (0,1), (2,3)) \\
&= + \det \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \\
&\quad - \det \begin{bmatrix} a_{00} & a_{02} \\ a_{10} & a_{12} \end{bmatrix} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \\
&\quad + \det \begin{bmatrix} a_{00} & a_{03} \\ a_{10} & a_{13} \end{bmatrix} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
&\quad + \det \begin{bmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{bmatrix} \det \begin{bmatrix} a_{20} & a_{23} \\ a_{30} & a_{33} \end{bmatrix} \\
&\quad - \det \begin{bmatrix} a_{01} & a_{03} \\ a_{11} & a_{13} \end{bmatrix} \det \begin{bmatrix} a_{20} & a_{22} \\ a_{30} & a_{32} \end{bmatrix} \\
&\quad + \det \begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \det \begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix}
\end{aligned} \tag{15}$$

The visualization for this approach, similar to that of Figure 4.1, is shown in Figure 4.2:

Figure 4.2 A visualization of the expansion by rows 0 and 1 of a 4×4 matrix in order to compute the determinant.



Computing the determinant of a 2×2 matrix requires 1 multiplication and 1 addition (or subtraction). The operation count is listed as a 2-tuple, the first component the number of multiplications and the second component the number of additions:

$$\Theta_2 = (2, 1)$$

Computing the determinant of a 3×3 matrix, when expanded by the first row according to Equation (8), requires the following number of operations

$$\Theta_3 = 3\Theta_2 + (3, 2) = (9, 5)$$

Using the row expansion of Equation (14) to compute the determinant of a 4×4 matrix, the operation count is

$$\Theta_4 = 4\Theta_3 + (4, 3) = (40, 23)$$

However, if you use Equation (15) to compute the determinant, the operation count is

$$\Theta'_4 = 12\Theta_2 + (6, 5) = (30, 17)$$

The total number of operations using Equation (14) is 63 and the total number of operation using Equation (15) is 47, so the latter equation is more efficient in terms of operation count.

To compute the inverse of a 4×4 matrix A, construct the adjugate matrix, which is the transpose of the matrix of cofactors for A. The cofactors involve 3×3 determinants. For example, the entry in row 0 and column 0 of $\text{adj}(A)$ is

$$+\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = +a_{11} \cdot \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

This equation involves determinants of 2×2 submatrices that also occur in the equation for the determinant of the 4×4 matrix. This suggests computing all of the entries of $\text{adj}(A)$ using only 2×2 submatrices.

Specifically, define

$$s_0 = \det \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}, \quad c_5 = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$s_1 = \det \begin{bmatrix} a_{00} & a_{02} \\ a_{10} & a_{12} \end{bmatrix}, \quad c_4 = \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$s_2 = \det \begin{bmatrix} a_{00} & a_{03} \\ a_{10} & a_{13} \end{bmatrix}, \quad c_3 = \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$s_3 = \det \begin{bmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{bmatrix}, \quad c_2 = \det \begin{bmatrix} a_{20} & a_{23} \\ a_{30} & a_{33} \end{bmatrix}$$

$$s_4 = \det \begin{bmatrix} a_{01} & a_{03} \\ a_{11} & a_{13} \end{bmatrix}, \quad c_1 = \det \begin{bmatrix} a_{20} & a_{22} \\ a_{30} & a_{32} \end{bmatrix}$$

$$s_5 = \det \begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix}, \quad c_0 = \det \begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix}$$

Then

$$\det(A) = s_0 c_5 - s_1 c_4 + s_2 c_3 + s_3 c_2 - s_4 c_1 + s_5 c_0$$

and

$$\text{adj}(A) = \begin{bmatrix} +a_{11}c_5 - a_{12}c_4 + a_{13}c_3 & -a_{01}c_5 + a_{02}c_4 - a_{03}c_3 & +a_{31}s_5 - a_{32}s_4 + a_{33}s_3 & -a_{21}s_5 + a_{22}s_4 - a_{23}s_3 \\ -a_{10}c_5 + a_{12}c_2 - a_{13}c_1 & +a_{00}c_5 - a_{02}c_2 + a_{03}c_1 & -a_{30}s_5 + a_{32}s_2 - a_{33}s_1 & +a_{20}s_5 - a_{22}s_2 + a_{23}s_1 \\ +a_{10}c_4 - a_{11}c_2 + a_{13}c_0 & -a_{00}c_4 + a_{01}c_2 - a_{03}c_0 & +a_{30}s_4 - a_{31}s_2 + a_{33}s_0 & -a_{20}s_4 + a_{21}s_2 - a_{23}s_0 \\ -a_{10}c_3 + a_{11}c_1 - a_{12}c_0 & +a_{00}c_3 - a_{01}c_1 + a_{02}c_0 & -a_{30}s_3 + a_{31}s_1 - a_{32}s_0 & +a_{20}s_3 - a_{21}s_1 + a_{22}s_0 \end{bmatrix}$$

If the determinant is not zero, then the inverse of A is computed using Equation (7).